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LOCAL MONODROMY IN NON-ARCHIMEDEAN ANALYTIC GEOMETRY

LORENZO RAMERO

fifth release

CONTENTS

Lorenzo Ramero
Université de Bordeaux I
Institut de Mathématiques
351, cours de la Liberation
F-33405 Talence Cedex
e-mail address: `ramero@math.u-bordeaux.fr`
web page: `http://www.math.u-bordeaux.fr/~ramero`

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0. Introduction (Snapshots from an Expedition)

0.1. **Where we came from.** Let $F := k((T))$ be the field of Laurent power series with coefficients in a field k; if k has positive characteristic, denote by G_F the Galois group of a separable closure of F. As it is well known, every ℓ-adic representation admits a natural *break decomposition* as a finite direct sum of $\mathbb{Q}_\ell[G_F]$-submodules. These decompositions are nicely compatible with the standard operations defined on the category $\mathbb{Q}_\ell[G_F]$-$\mathbf{Mod}$ of $\mathbb{Q}_\ell[G_F]$-modules, such as tensor products and Hom functors.

On the other hand, if k has characteristic zero, one may consider the category $\mathsf{D.E.}(F/k)$ of finite dimensional F-vector spaces endowed with a T-adically continuous k-linear connection. Then the theory of Levelt-Turritin says that every object of $\mathsf{D.E.}(F/k)$ admits a natural decomposition, satisfying wholly analogous compatibilities (see [33]).

The parallelisms revealed by the study of $\mathbb{Q}_\ell[G_F]$-$\mathbf{Mod}$ and $\mathsf{D.E.}(F/k)$ are aboundant and striking, to the point that one can write down a heuristic dictionary to translate definitions and theorems back and forth between them (see [33, Appendix]). More recently, Yves André has introduced notions of *slope filtration* and *Hasse-Arf filtration* for a general tannakian category (see [1, Déf.1.1.1 and Déf.2.2.1]), which extract the basic features that are common to these two categories (and indeed, to others as well). He has shown that, quite generally, the existence of a Hasse-Arf filtration imposes very binding restrictions on the structure of a tannakian category.

0.2. **Where we were heading.** The aim of this work is to exhibit another specimen of the same sort as those considered by André. Namely, let $(K, |\cdot|)$ be an algebraically closed valued field of mixed characteristic $(0, p)$, complete for its rank one valuation $|\cdot| : K \to \Gamma_K \cup \{0\}$ (we may view the value group Γ_K as a subgroup of $\mathbb{R}_{>0}$). Let also Λ be a local ring which is a filtered union of finite rings on which p is invertible. Our objects of study are the locally constant sheaves of free Λ-modules of finite rank on the étale site of the punctured disc :

$$\mathbb{D}(r)^* := \{x \in K \mid 0 < |x| \leq r\} \qquad \text{(for any } r \in \Gamma_K\text{)}.$$

These modules form a category Λ-$\mathbf{Loc}(r)$, which is tannakian if Λ is a field. However, we are really interested in describing the *local monodromy* of these sheaves, *i.e.* their behaviour in an arbitrarily small neighborhood of the missing center of the disc, hence we do not distinguish two local systems $\mathscr{F}$ and $\mathscr{F}'$ on the étale sites $\mathbb{D}(r)^*_{\text{ét}}$, respectively $\mathbb{D}(r')^*_{\text{ét}}$, if they become isomorphic after restriction to some smaller disc $\mathbb{D}(r'')^*$ (with $0 < r'' \leq r, r'$). Hence, we are really concerned with the 2-colimit category :

$$\Lambda\text{-}\mathbf{Loc}(0^+) := \operatorname*{colim}_{r \to 0^+} \Lambda\text{-}\mathbf{Loc}(r).$$

0.3. **What we hoped to find there.** It is instructive to consider first the case where the monodromy is finite, *i.e.* the local system $\mathscr{F}$ under consideration becomes constant on some finite Galois étale covering $X \to \mathbb{D}(r)^*$, say with Galois group G. This case is already highly non-trivial : $\mathscr{F}$ is the same as a $\Lambda[G]$-module of finite type, and if we insisted on a complete description of the *global* monodromy of $\mathscr{F}$, we would have to classify all such representations, so essentially all possible finite coverings of $\mathbb{D}(r)^*$ – a task that is probably beyond the reach of current techniques. On the other hand, the *germs* of finite coverings of $\mathbb{D}(r)^*$ are completely classified by the so-called p-adic Riemann existence theorem, proved by O.Gabber around 1982 (unpublished) and by W.Lütkebohmert about ten years later ([40]). Explicitly, after restriction to a smaller disc, every such finite covering becomes a disjoint union of cyclic coverings of

Kummer type; therefore the local monodromy of our $\mathscr{F}$ will be a representation of a finite cyclic group.

Though no such description is known for general étale coverings of $\mathbb{D}(1)^*$ (*i.e.* for those of infinite degree), this provides some evidence for the thesis that, by shifting the focus to germs of coverings, one should expect to reach a new, but substantially tamer mathematical territory – one in which some general geographical features are discernible and can be used as worthwhile reference points.

0.4. **Why we were not disappointed.** This expectation is largely borne out by our main theorem 4.2.42, which can be stated as follows. Suppose that $H^1(\mathbb{D}(r)^*_{\text{ét}}, \mathscr{F})$ is a Λ-module of finite type, in which case we say that $\mathscr{F}$ has *bounded ramification*; this condition is independent of the value $r \in \Gamma_K$. Then there exists a connected open subset $U \subset \mathbb{D}(r)^*$ such that $U \cap \mathbb{D}(\varepsilon)^* \neq \varnothing$ for every $\varepsilon > 0$, and such that the restriction of F to $U_{\text{ét}}$ admits a *break decomposition* as a direct sum of locally constant subsheaves :

$$\mathscr{F}_{|U} \simeq \bigoplus_{\gamma \in \Gamma_0} \mathscr{F}(\gamma) \tag{0.4.1}$$

indexed by the ordered group Γ_0, which is the product of ordered groups $\mathbb{Q} \times \mathbb{R}$, endowed with the lexicographic ordering (of course $F(\gamma) \neq 0$ for only finitely many $\gamma \in \Gamma$). This decomposition is compatible in the usual way with tensor products and Hom functors; moreover, one may define Swan conductors for $\mathscr{F}$, and there is also an adequate analogue of the Hasse-Arf theorem. Since the Swan conductor determines the Euler-Poincaré characteristic of $R\Gamma(\mathbb{D}(r)^*_{\text{ét}}, \mathscr{F})$, it follows easily that, in case Λ is a field, the subcategory Λ-$\mathbf{Loc}(0^+)_{\text{b.r.}}$ of Λ-$\mathbf{Loc}(0^+)$ consisting of Λ-modules with bounded ramification, is tannakian. Therefore, the break decomposition in Λ-$\mathbf{Loc}(0^+)_{\text{b.r.}}$ would be precisely what is needed to define a filtration of Hasse-Arf type, in the sense of [1], if it were not for the following two short-comings. First, the filtration is not indexed by the real numbers, but by the more complicated group Γ_0. This is however a minor divergence, which can be cured, for instance, by generalizing a little the definition of slope filtration in a tannakian category. More seriously, the break decomposition of an object of Λ-$\mathbf{Loc}(0^+)_{\text{b.r.}}$ is defined (a priori) only in a strictly larger tannakian category; this is because the open subset U may not contain any punctured disc $\mathbb{D}(\varepsilon)^*$ (though it intersects all of them).

I expect that actually every local system with bounded ramification admits a break decomposition already over some small punctured disc, and I hope to address this question in a future release of this work. Once this result is available, it will be possible to apply the tannakian machinery of [1] to study the structure of such local systems.

0.5. **Planning for the journey.** The proof of theorem 4.2.42 is divided into two separate steps. The first step consists in describing the monodromy of $\mathscr{F}$ *around the border* of a disc $\mathbb{D}(r)^*$. This is one of the points where the standard topological intuition may be misleading: a non-archimedean punctured disc is far from being "homotopically equivalent" to an annulus, for any reasonable notion of homotopy equivalence. Indeed, it is easy to construct examples of (finite or infinite) connected Galois coverings of $\mathbb{D}(r)^*$ that are completely split at the border, that is over every annulus of the form:

$$\mathbb{D}(a, r) := \{x \in K \mid a \leq |x| \leq r\}$$

with $a \in \Gamma_K$ sufficiently close to r. (And conversely, an étale covering of $\mathbb{D}(r)^*$ may be completely split near the center, and connected on every annulus $\mathbb{D}(a, r)$ with a sufficiently close to r.) But the crucial difference is that the monodromy around the border is *always finite*. As a consequence, the study of this monodromy is an essentially algebraic affair that can be carried out by a suitable extension of classical ramification theory for henselian discrete valuation rings with perfect residue field.

This extension has been developed by R.Huber in [31]: the main tool is a certain rank two valuation $\eta(r)$, defined on the ring of analytic functions $\mathscr{O}(\mathbb{D}(a,r))$ on $\mathbb{D}(a,r)$ (for any $a<r$), and continuous for the p-adic topology. In a precise sense (best expressed in the language of adic spaces), this valuation is localized at the border of the disc. Moreover, the value group $\Gamma_{\eta(r)}$ of $\eta(r)$ is naturally isomorphic to $\Gamma_K\times\mathbb{Z}$, ordered lexicographically. Now, let $f:X\to\mathbb{D}(a,r)_{\text{ét}}$ be a finite, connected, étale and Galois covering of Galois group G, such that $f^*\mathscr{F}$ is a constant sheaf. Then $\eta(r)$ admits finitely many extensions $x_1,\dots,x_n$ to $\mathscr{O}(X)$, and the action of G permutes transitively these extensions. Fix one of these valuations $x:=x_i$; the stabilizer $St_x\subset G$ can be naturally identified with the Galois group of a finite separable extension

$$\kappa(\eta(r))^{\wedge h}\subset\kappa(x)^{\wedge h}$$

of henselian valued fields, obtained by suitably henselizing the completions (for the valuation topologies) $\kappa(\eta(r))^\wedge$, $\kappa(x)^\wedge$ of the fields of fractions of $\mathscr{O}(\mathbb{D}(a,r))$ and respectively $\mathscr{O}(X)$. Let $\pi_1(r)$ be the Galois group of a separable closure of $\kappa(\eta(r))^{\wedge h}$; we may regard $\mathscr{F}$ as a $\Lambda[G]$-module, hence as a $\Lambda[St_x]$-module, by restriction, and then as a $\Lambda[\pi_1(r)]$-module, by inflation.

The group $\pi_1(r)$ admits a natural (upper numbering) higher ramification filtration, wholly analogous to the standard one for discrete valued fields, except that it is indexed by the ordered group $\Gamma_{\eta(r)}\otimes_{\mathbb{Z}}\mathbb{Q}$. Therefore, when Λ is a field, the tannakian category $\Lambda[\pi_1(r)]$-$\mathbf{Mod}$ is yet another example of a category with a slope filtration, except that the filtration is indexed by $\Gamma_{\eta(r)}\otimes_{\mathbb{Z}}\mathbb{Q}$, rather than by $\mathbb{R}$. Moreover, this slope filtration is even of *Hasse-Arf type*, provided we redefine appropriately the Newton polygon of a representation.

0.6. **Division of labour.** R.Huber's ramification theory yields, for every radius $r\in\Gamma_K$, a $\Lambda[\pi_1(r)]$-equivariant break decomposition of the stalk $\mathscr{F}_{\eta(r)}$. The second step of the proof of theorem 4.2.42 consists in describing how this decomposition evolves as r changes. This step presents in turn two aspects: on the one hand, we have to examine the *continuity* properties of the breaks, *i.e.* the way the decomposition varies in a neighborhood of a given radius r; on the other hand, we have to make an *asymptotic* study, to determine the behaviour of the decomposition for $r\to 0^+$. The upshot is that, for large values of r, the breaks of $\mathscr{F}_{\eta(r)}$ vary in a continuous, but apparently patternless manner; but, as we approach the missing center of the disc, eventually a coherent order emerges: the decompositions fall into step, and they give rise to the asymptotic decomposition (0.4.1).

0.7. **Surveyor's gear.** Both the continuity and the aymptotic study ultimately rely on the remarkable properties of certain conductor functions attached to our local system $\mathscr{F}$. To define these conductors we may assume, without loss of generality, that $\Gamma_K=\mathbb{R}_{>0}$. Suppose that $f:X\to\mathbb{D}(a,b)$ is a finite Galois étale covering, with group G, such that $f^*\mathscr{F}$ is constant. For every $r\in[a,b]$, we have the $\pi_1(r)$-equivariant break decomposition

$$\mathscr{F}_{\eta(r)}\simeq M_1(\gamma_1(r))\oplus\cdots\oplus M_n(\gamma_n(r))$$

where n depends also on r, and the breaks $\gamma_i(r)$ live in $\mathbb{R}_{>0}\times\mathbb{Q}$. For any $\gamma\in\mathbb{R}_{>0}\times\mathbb{Q}$, let us denote by $\gamma^\flat$ and $\gamma^\natural$ the projections of γ on $\mathbb{R}_{>0}$ and respectively $\mathbb{Q}$. Set also $m_i:=\mathrm{rk}_\Lambda M_i(\gamma_i(r))$ for every $i=1,\dots,n$. Then we may consider the conductor functions :

$$\delta_{\mathscr{F}}:[\log 1/b,\log 1/a]\to\mathbb{R}_{\geq 0}\quad\text{and}\quad \mathsf{sw}^\natural(\mathscr{F},\cdot):[a,b]\to\mathbb{Z}$$

defined by letting :

$$\delta_{\mathscr{F}}(-\log r):=-\sum_{i=1}^n\log\gamma_i(r)^\flat\cdot m_i\quad\text{and}\quad \mathsf{sw}^\natural(\mathscr{F},r^+):=\sum_{i=1}^n\gamma_i(r)^\natural\cdot m_i.$$

We show that $\delta_{\mathscr{F}}$ is a piecewise linear, continuous and convex function, and moreover the right derivative of $\delta_{\mathscr{F}}$ is computed by $\mathsf{sw}^\natural(\mathscr{F},\cdot)$ (see proposition 4.1.8). The function $\mathsf{sw}^\natural(\mathscr{F},\cdot)$ can

be characterized in terms of the Swan conductor of the covering X. Namely, for every $r \in [a,b]$, choose a valuation x of $\mathscr{O}(X)$ extending the valuation $\eta(r)$; then the higher ramification filtration of St_x determines, in the usual way, a $\mathbb{Z}$-valued Swan character sw_x of St_x (see 3.3), which is the character of an element of $K^0(\mathbb{Z}_\ell[St_x])$. We induce to get a virtual character of G :

$$\mathsf{sw}^\natural_G(r^+) := \mathrm{Ind}^G_{St_x}\mathsf{sw}^\natural_x$$

which is independent of the choice of x. Let now $\rho \in K_0(\Lambda[G])$ be the $\Lambda[G]$-module corresponding to $\mathscr{F}$; since $\mathsf{sw}^\natural_G(r^+)$ lies in $K^0(\mathbb{Z}_\ell[G])$, we may apply the natural pairing

$$\langle\cdot,\cdot\rangle_G : K^0(\mathbb{Z}_\ell[G]) \times K_0(\Lambda[G]) \to \mathbb{Z}$$

and we obtain the identity :

$$\mathsf{sw}^\natural(\mathscr{F}, r^+) = \langle \mathsf{sw}^\natural_G(r^+), \rho\rangle_G.$$

On the other hand, for every $r \in [a,b]$ one can consider the preimage $X(r) \subset X$ of the annulus $\mathbb{D}(r,r) \subset \mathbb{D}(a,b)$. The ring of analytic functions $\mathscr{O}^+_X(r)$ on $X(r)$ whose sup-norm is ≤ 1 is a finite free module over the analogous ring $\mathscr{O}^+_{\mathbb{D}}(r)$ of bounded functions on $\mathbb{D}(r,r)$. Hence the discriminant $\mathfrak{d}^\flat_f(r)$ of this ring extension is well-defined, and it is an invertible function on $\mathbb{D}(r,r)$, since f is étale. Its sup-norm $|\mathfrak{d}^\flat_f(r)|$ is a real number in $]0,1]$, and $\delta_f(-\log r) := -\log|\mathfrak{d}^\flat_f(r)| \in \mathbb{R}_{\geq 0}$. Let now $\mathscr{G} := f_*\Lambda_X$ be the direct image of the constant sheaf Λ_X on $X_{\text{ét}}$; then $\mathscr{G}$ corresponds to the regular representation of G, and we have the identity :

$$\delta_{\mathscr{G}} = \delta_f.$$

Hence, the right (logarithmic) derivative of the discriminant is the Swan conductor of the regular representation of G : this is our analogue of Hasse's Führerdiskriminantenproduktformel.

0.8. **Detour to visit a relative.** The proof of the convexity of $\delta_{\mathscr{F}}$ is accomplished by a rather technical argument, involving semi-stable reduction and a vanishing cycle calculation. As a corollary, one derives a proof of the convexity of the discriminant function δ_f. However, the convexity of δ_f can also be shown by a completely elementary argument that uses little more than some valuation theory, the first rudiments of the theory of adic spaces, and some simple tools from p-adic analysis borrowed from [6] and the first chapter of [24]. We present this argument in section 2.3, since it is of independent interest : indeed, as explained in section 2.4, with its aid one may quickly derive a proof of the p-adic Riemann existence theorem. This proof is not only much more elementary than Lütkebomert's; it is also significantly simpler than Gabber's original argument[1]. All in all, I believe it is a convincing demonstration of the new possibilities opened up by the theory of adic spaces.

0.9. **Dulcis in fundo.** The ideas which enable to tackle successfully the asymptotic study of the break are developed in section 4.2, and find their roots in harmonic analysis techniques, such as Fourier transform and the allied method of stationary phase; this should come as no surprise to any reader familiar with the works of Katz (*e.g.* [35]) or Laumon ([38]). Closer to home, these ideas represent an extension (perhaps, a vindication) of my previous work [42], where I introduced a Fourier transform for sheaves of Λ-modules on the étale site of the analytification $(\mathbb{A}^1_K)^{\mathrm{ad}}$ of the affine line. For more details, we refer the reader to remark 4.2.18. This intrusion of concepts and viewpoints originating from such a seemingly far removed area of mathematics, is for me one of the most appealing aspects of this project. It was already one of the main themes in [42], and I believe that it runs deeper than a mere formal analogy : for instance, from this perspective, formula (0.4.1) is none else than a spectral decomposition of the local system $\mathscr{F}$. Whereas [42] dealt only with a suggestive, but *ad hoc* class of local systems, we have now a good grasp of all those local systems $\mathscr{F}$ whose ramification is bounded. This boundedness

[1] Of course, the reader will have to take my word for it, since Gabber never published his proof.

condition can also be restated in terms of Swan conductors, hence it is, on the one hand a purely local condition that serves to circumscribe *the* good class of local systems that should be stable under the usual Yoga of cohomological operations. On the other hand, it is a finiteness condition on the cohomology of $\mathscr{F}$, hence – from the harmonic viewpoint – it is essentially like confining our attention to the class of "integrable sheaves"; *i.e.* we are really doing harmonic analysis in the space L_1 : a most natural restriction.

We cannot resist ending on a more speculative note. As it has been seen, in many situations local monodromy is described via the higher ramification filtration on a Galois group, defined by an appropriate valuation. This cannot be literally true for the local monodromy theory of the punctured disc, since the trivializing covering $X \to \mathbb{D}(r)^*$ of a local system may have infinite degree, in which case the field of fractions of $\mathscr{O}(X)$ has infinite transcendence degree over the field of fractions $\mathscr{O}(\mathbb{D}(r)^*)$. Nevertheless, one may ask whether there exists a valuation "localized at the origin", which governs, in some mysterious way, the local monodromy theory of $\mathbb{D}(r)^*$. It turns out that there exists one natural candidate, which is well-defined on the ring $A := K[T, T^{-1}]$ (of regular functions on the "algebraic punctured disc") : namely, the rank two specialization w of the degree valuation $v : A \to \mathbb{Z} \cup \{\infty\}$ such that $v(T^n) := n$ for every $n \in \mathbb{Z}$. The value group of w is the lexicographically ordered group $\Delta := \mathbb{Z} \times \Gamma_K$, and one has the rule : $w(a \cdot T^n) := (n, |a|)$, for every $n \in \mathbb{Z}$ and every $a \in K$. Notice that $\Delta \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic – as an ordered group – to the group Γ_0 that indexes our break decomposition. However, this valuation w is not p-adically continuous, hence it does not lie in the adic spectrum $\mathrm{Spa}\, A$, but only in the larger valuation spectrum $\mathrm{Spv}\, A$.

Acknowledgements: I am hugely indebted with Ofer Gabber for explaining me the proof of his unpublished theorem and for many discussions, through which I have learned many of the techniques that play an essential role in this work. Especially, I have learned from him an idea, due originally to Deligne ([37]), that allows to calculate the rank of vanishing cycles; the same idea is recycled in the proof of proposition 3.2.30. I thank Isabelle Vidal for sending me a copy of her thesis [48], where she uses de Jong's method of alterations to deduce consequences for the étale cohomology of schemes; her argument is applied here to the study of vanishing cycles (theorem 3.2.17). I also wholeheartedly acknowledge the support of the IHES and the Max-Planck Institute in Bonn, where large parts of this research have been carried out. This paper was stimulated and inpired by Roland Huber's work [31].

1. Algebraic preliminaries

1.1. **Power-multiplicative seminorms.** Real-valued valuations on fields and topological algebras have been standard tools in p-adic analytic geometry since the earliest infancy of the subject; by contrast, the role played by higher rank valuations in several fundamental questions has been recognized only in recent times.

In non-archimedean analysis one encounters more generally certain ultrametric real-valued norms (or seminorms) that are not multiplicative, but only power-multiplicative. We shall see that higher rank power-multiplicative seminorms appear just as naturally, and are just as useful.

1.1.1. In this section we let $(\Gamma, <)$ be a totally ordered abelian group, whose neutral element is denoted by 1 and whose composition law we write multiplicatively. As customary, we shall extend the ordering and the composition law of Γ to the set $\Gamma \cup \{0\}$, in such a way that $\gamma > 0$ for every $\gamma \in \Gamma$, and $\gamma \cdot 0 = 0 \cdot \gamma = 0$ for every $\gamma \in \Gamma \cup \{0\}$. Notice also that the ordering of Γ extends uniquely to $\Gamma_{\mathbb{Q}} := \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$. Finally we let $\Gamma^+ := \{\gamma \in \Gamma \mid \gamma \leq 1\}$

Definition 1.1.2. Let A be a ring. A *power-multiplicative* Γ*-valued seminorm* on A is a map

$$|\cdot| : A \to \Gamma \cup \{0\}$$

satisfying the following conditions:

(a) $|0| = 0$ and $|1| = 1$.
(b) $|x - y| \leq \max(|x|, |y|)$ for all $x, y \in A$.
(c) $|x \cdot y| \leq |x| \cdot |y|$ for all $x, y \in A$.
(d) $|x^n| = |x|^n$ for every $x \in A$ and every $n \in \mathbb{N}$.

One says also that $(A, |\cdot|)$ is a Γ-*seminormed ring*. If $|x| \neq 0$ whenever $x \neq 0$, one says that $|\cdot|$ is a Γ-*valued norm* and correspondingly one talks of Γ-normed rings. If in (c) we have equality for every $x, y \in A$, then we say that $(A, |\cdot|)$ is a Γ-*valued ring* and that $|\cdot|$ is a Γ-*valued valuation*. A morphism $\phi : (A, |\cdot|) \to (A', |\cdot|')$ of Γ-seminormed rings is a ring homomorphism $\phi : A \to A'$ such that $|\phi(a)|' = |a|$ for every $a \in A$. Notice that the subset

$$A^+ := \{a \in A \mid |a| \leq 1\} \subset A \tag{1.1.3}$$

is a subring of A. The *support* of $|\cdot|$ is the ideal

$$\mathrm{supp}(|\cdot|) := \{a \in A \mid |a| = 0\} \subset A.$$

If $|\cdot|$ is a valuation, $\mathrm{supp}(|\cdot|)$ is a prime ideal.

Lemma 1.1.4. *Let $(A, |\cdot|)$ be a seminormed ring and $x, y \in A$ any two elements such that $|x| \neq |y|$. Then $|x + y| = \max(|x|, |y|)$.*

Proof. Let us say that $|x| < |y|$. By (b) of definition 1.1.2 we have:

$$|y| \leq \max(|x + y|, |x|) \leq \max(|y|, |x|) = |y|.$$

Hence $|y| = |x + y|$, which is the claim. $\square$

1.1.5. Let $(A, |\cdot|)$ be a semi-normed ring. For every monic polynomial $p(T) := T^m + a_1T^{m-1} + \cdots + a_m \in A[T]$ we set

$$\sigma(p) := \max(|a_i|^{1/i} \mid i = 1, \dots, m) \in \Gamma_{\mathbb{Q}} \cup \{0\}$$

and call $\sigma(p)$ the *spectral value* of $p(T)$.

Lemma 1.1.6. *Let $p, q \in A[T]$ be monic polynomials. Then $\sigma(pq) \leq \max(\sigma(p), \sigma(q))$. If $|\cdot|$ is a valuation, the above inequality is, in fact, an equality.*

Proof. Mutatis mutandis, this is the same as the proof of [6, §1.5.4, Prop.1]. $\square$

1.1.7. Let A be a normal domain, K the field of fractions of A, and $A \to B$ an injective integral ring homomorphism such that B is torsion-free as an A-module. For any $b \in B \otimes_A K$ the set of polynomials $P(T) \in K[T]$ such that $P(b) = 0$ is an ideal, whose monic generator $\mu_b(T)$ is the *minimal polynomial* of b.

Lemma 1.1.8. *Keep the assumptions of* (1.1.7) *and suppose that $b \in B$. Then $\mu_b(T) \in A[T]$.*

Proof. By [41, Th.10.4] we have $A = \bigcap_v A_v$ where v ranges over all the valuations of K whose valuation ring A_v contains A. We can therefore suppose that A is the valuation ring of one such $v : A \to \Gamma_v \cup \{0\}$. For any polynomial $p(T) := \sum_{i=0}^n a_iT^i \in K[T]$ let

$$|p|_v := \max(v(a_i) \mid i = 0, \dots, n) \in \Gamma_v \cup \{0\}.$$

By assumption, there is a monic polynomial $P(T) \in A[T]$ such that $P(b) = 0$, hence μ_b divides P in $K[T]$. The assertion is therefore a consequence of the following:

Claim 1.1.9. $|p \cdot q|_v = |p|_v \cdot |q|_v$ for all $p, q \in K[T]$.

Proof of the claim. We leave it as an exercise for the reader: one can easily adapt the direct argument used in the classical proof of the Gauss lemma (cp. [6, p.44]). $\square$

1.1.10. In the situation of (1.1.7), let $b \in B$ and say that

$$\mu_b(T) = T^n + a_1 T^{n-1} + a_2 T^{n-2} + \cdots + a_n$$

for some $n \in \mathbb{N}$ and elements $a_1, \dots, a_n \in A$, in view of lemma 1.1.8. Suppose now that $|\cdot| : A \to \Gamma \cup \{0\}$ is a power multiplicative Γ-valued seminorm on A. Then the *spectral seminorm* of b is defined as

$$|b|_{\mathrm{sp}} := \sigma(\mu_b(T)) \in \Gamma_{\mathbb{Q}} \cup \{0\}.$$

The name of $|\cdot|_{\mathrm{sp}}$ is justified by the following:

Proposition 1.1.11. *With the notation of* (1.1.10)*, the pair* $(B, |\cdot|_{\mathrm{sp}})$ *is a* $\Gamma_{\mathbb{Q}}$*-seminormed ring.*

Proof. We consider the A-algebra $A' := A[\Gamma_{\mathbb{Q}}]$. Hence A' is generated as an A-algebra by elements $[\gamma]$, for all $\gamma \in \Gamma_{\mathbb{Q}}$, subject to the relations

$$[\gamma] \cdot [\delta] = [\gamma \cdot \delta] \qquad \text{for all } \gamma, \delta \in \Gamma_{\mathbb{Q}}.$$

Clearly, every element of A' admits a unique expression of the form $\sum_{\gamma \in \Gamma_{\mathbb{Q}}} a_\gamma \cdot [\gamma]$ where $a_\gamma = 0$ for all but finitely many $\gamma \in \Gamma_{\mathbb{Q}}$. Set also $B' := B \otimes_A A'$.

Claim 1.1.12. A' is a normal domain, flat over A, and B' is a torsion-free A'-module.

Proof of the claim. We can write $\Gamma_{\mathbb{Q}} = \bigcup_{i \in I} \Gamma_i$, the filtered union of all the finitely generated subgroups $\Gamma_i \subset \Gamma_{\mathbb{Q}}$. Then $A' = \bigcup_{i \in I} A[\Gamma_i]$, and it suffices to show the claim for the subalgebras $A'_i := A[\Gamma_i]$. Since Γ is torsion-free, we have (non-canonical) isomorphisms $\Gamma_i \simeq \mathbb{Z}^{N_i}$, whence $A'_i \simeq A[T_1^{\pm 1}, \dots, T_N^{\pm 1}] \simeq A \otimes_{\mathbb{Z}} \mathbb{Z}[T_1^{\pm 1}, \dots, T_N^{\pm 1}]$, from which flatness is clear. Normality follows as well, in view of [20, Ch.IV, Prop.6.14.1]. Likewise, B' is the filtered union of the A'_i-algebra $B[T_1^{\pm 1}, \dots, T_N^{\pm 1}]$, and the latter are clearly torsion-free over A'_i. $\Diamond$

We define a map $|\cdot| : A' \to \Gamma_{\mathbb{Q}}$ by the rule:

$$\sum_{\gamma \in \Gamma_{\mathbb{Q}}} a_\gamma \cdot [\gamma] \mapsto \max(|a_\gamma| \cdot \gamma \mid \gamma \in \Gamma_{\mathbb{Q}}).$$

Claim 1.1.13. $(A', |\cdot|)$ is a $\Gamma_{\mathbb{Q}}$-seminormed ring.

Proof of the claim. All conditions of definition 1.1.2 are clearly fulfilled, except possibly for (d). However, for given $x = \sum_\gamma a_\gamma \cdot [\gamma]$, let $\delta \in \Gamma$ be the minimal element such that $|a_\delta| \cdot \delta = |x|$. Say that $x^n = \sum_\gamma b_\gamma \cdot [\gamma]$; we have

$$b_{\delta^n} \cdot [\delta^n] = \sum_{\gamma_1 \cdot \ldots \cdot \gamma_n = \delta^n} a_{\gamma_1} \cdot [\gamma_1] \cdot \ldots \cdot a_{\gamma_n} \cdot [\gamma_n].$$

If now $\gamma_1 \cdot \ldots \cdot \gamma_n = \delta^n$ and the γ_i are not all equal to δ, then necessarily $\gamma_i < \delta$ for some $i \leq n$. By the choice of δ it follows that $|a_{\gamma_i}| \cdot \gamma_i < |a_\delta| \cdot \delta$, hence $|a_{\gamma_1} \cdot [\gamma_1] \cdot \ldots \cdot a_{\gamma_n} \cdot [\gamma_n]| < |x|^n$. In view of lemma 1.1.4 we deduce that $|b_{\delta^n}| \cdot \delta^n = |x|^n$, so (d) holds as well. $\Diamond$

From claim 1.1.12 it follows that the induced map $A' \to B'$ is injective, and by claim 1.1.13 we can replace A, B and Γ by respectively A', B' and $\Gamma_{\mathbb{Q}}$, which allows us to assume that

(1.1.14) $|A| = |B|_{\mathrm{sp}} = \Gamma \cup \{0\} = \Gamma_{\mathbb{Q}} \cup \{0\}$ and there is a group homomorphism $[\cdot] : \Gamma \to A^\times$ which is a left inverse for $|\cdot| : A \to \Gamma \cup \{0\}$.

Let $B^+ := \{x \in B \mid |x|_{\mathrm{sp}} \leq 1\}$.

Claim 1.1.15. A^+ is normal and B^+ is the integral closure of A^+ in B.

Proof of the claim. To show that A^+ is normal, it suffices to prove that A^+ is integrally closed in A. Hence, suppose that $x \in A$ satisfies an equation of the form $x^n + a_1 x^{n-1} + \cdots + a_n = 0$, where $|a_i| \leq 1$ for $i = 1, \dots, n$. It follows that $|x|^n \leq \max(|x|^{n-i} \mid i = 1, \dots, n)$, which is possible only when $|x| \leq 1$, as required. Next, let $b \in B^+$; by definition this means that $\mu_b(T) \in A^+[T]$, so x is integral over A^+. Conversely, we apply lemma 1.1.8 with A^+ in place of A, to see that $|b|_{\mathrm{sp}} \leq 1$ for every $b \in B$ which is integral over A^+. ◊

Finally, we verify conditions (a)-(d) of definition 1.1.2. (a) is obvious. Let $x, y \in B$ and say that $\delta := |x| \leq \gamma := |y|$; thanks to (1.1.14) we have

$$|x \cdot [\gamma^{-1}]| \leq |x| \cdot \gamma^{-1} \leq 1$$

and likewise $|y \cdot [\gamma^{-1}]| \leq 1$. By claim 1.1.15 it follows that $x \cdot [\gamma^{-1}]$ and $y \cdot [\gamma^{-1}]$ are integral over A^+, hence the same holds for their sum and again claim 1.1.15 implies that $|(x+y) \cdot [\gamma^{-1}]| \leq 1$. Consequently $|x + y| \leq |(x + y) \cdot [\gamma^{-1}]| \cdot \gamma \leq |y|$, which is (b). For (c) one considers the product $x \cdot [\delta^{-1}] \cdot y \cdot [\gamma^{-1}]$ which is integral over A^+ by an analogous argument; then $|x \cdot y| \leq |x \cdot [\delta^{-1}] \cdot y \cdot [\gamma^{-1}]| \cdot \delta \cdot \gamma \leq |x| \cdot |y|$, which is (c). Finally, suppose that $|x^n| = \varepsilon < \delta^n$; by (1.1.14) the value group Γ is divisible, hence we can consider the element $z := x \cdot [\varepsilon^{-1/n}]$ and in fact $|z^n| \leq 1$, hence z^n is integral over A^+, so the same holds for z, and again $|z| \leq 1$, therefore $|x| \leq \varepsilon^{1/n} < \delta$, a contradiction that shows (d). □

Remark 1.1.16. (i) Proposition 1.1.11 generalizes [6, §3.2.2, Th.2], which deals with the special case of real-valued norms. The proof of *loc.cit.* does not extend to the present case, since it is based on a smoothing technique that makes sense only for real-valued seminorms.

(ii) In the situations encountered in later sections, it is probably not too hard to verify directly that the spectral norm is power-multiplicative (the same can already be said for most applications of [6, §3.2.2, Th.2]). However, it seems desirable to have a general statement such as proposition 1.1.11.

Lemma 1.1.17. *In the situation of* (1.1.10) *:*

- (i) *Let* $(A, |\cdot|) \to (A', |\cdot|')$ *be a flat morphism of* Γ*-seminormed normal domains, suppose that* $B' := A' \otimes_A B$ *is torsion-free over* A'*, and endow it with the spectral seminorm* $|\cdot|'_{\mathrm{sp}}$ *relative to the induced injective ring homomorphism* $A' \to B'$*. Then:*
  - (a) *The natural map* $(B, |\cdot|_{\mathrm{sp}}) \to (B', |\cdot|'_{\mathrm{sp}})$ *is a morphism of* $\Gamma_{\mathbb{Q}}$*-seminormed rings.*
  - (b) *If* $|\cdot| : A \to \Gamma \cup \{0\}$ *is a valuation, we have* $|ab|_{\mathrm{sp}} = |a| \cdot |b|_{\mathrm{sp}}$ *for every* $a \in A$ *and* $b \in B$*.*
- (ii) *Suppose that* $B = B_1 \times \cdots \times B_r$*, where each* B_i *is an* A*-algebra fulfilling the conditions of* (1.1.7)*, and for* $i = 1, \dots, r$*, denote by* $|\cdot|_{\mathrm{sp},i}$ *the spectral norm of* B_i*. Then, for every* $b := (b_1, \dots, b_r) \in B$ *we have:*

$$|b|_{\mathrm{sp}} \leq \max(|b_i|_{\mathrm{sp},i} \mid i = 1, \dots, r)$$

  and if the norm of A *is a valuation, the inequality is actually an equality.*
- (iii) *Suppose that* $(A, |\cdot|) \to (B, |\cdot|_B)$ *is an extension of valuation rings, such that* B *is integral over* A*. Then the spectral norm* $|\cdot|_{\mathrm{sp}}$ *is a valuation equivalent to* $|\cdot|_B$*.*

Proof. (i.a): For given $b \in B$, let $C \subset B$ be a finite A-subalgebra with $b \in C$; then $C' := A' \otimes_A C \subset B'$. Let K and K' be the fields of fraction of A and respectively A'; the element b induces a K-linear (resp. K'-linear) endomorphisms on the finite dimensional K-vector space (resp. K'-vector space) $C \otimes_A K$ (resp. $C' \otimes_{A'} K'$), and the spectral seminorms of b in B and B' are defined in terms of the minimal polynomials of these endomorphisms. Hence the assertion boils down to the invariance of the minimal polynomial under base field extensions.

(i.b): If $\mu_b(T) = T^n + a_1 T^{n-1} + a_2 T^{n-2} + \cdots + a_n$, then $\mu_{ab}(T) = T^n + a \cdot a_1 T^{n-1} + a^2 \cdot a_2 T^{n-2} + \cdots + a^n \cdot a_n$, from which the assertion follows easily.

(ii): The minimal polynomial of b is the least common multiple of the minimal polynomials of $b_1, \dots, b_r$; hence the claim follows from lemma 1.1.6.

(iii): Let $b \in B$ such that $|b|_B = 1$, and $\mu_b(T) := T^n + \sum_{i=1}^n a_i T^{n-i}$ the minimal polynomial of b over $\mathrm{Frac}(A)$. Since b is integral over A, we have $a_i \in A$ for every $i = 1, \dots, n$, hence $|b|_{\mathrm{sp}} \leq 1$; on the other hand, since :

$$1 = |b|_B^n = |\sum_{i=1}^n a_i b^{n-i}|_B \leq \max(|a_i| \mid i = 1, \dots, n)$$

we have as well : $1 \leq |b|_{\mathrm{sp}}$, so that $|b|_{\mathrm{sp}} = 1$. Finally, for a general element $b \in B \setminus \{0\}$, we may find $s \in \mathbb{N} \setminus \{0\}$ and $a \in A$ such that $|b^s \cdot a|_B = 1$, hence $|b|_B^s \cdot |a| = 1 = |b|_{\mathrm{sp}}^s \cdot |a|$, in view of (i.b). The assertion follows. □

1.2. **Normed modules.** Throughout this section $(A, |\cdot|)$ denotes a Γ-normed ring (for some ordered abelian group Γ). Following ([6, §2.1.1, Def.1]), we shall say that a *faithfully Γ-seminormed A-module* is a pair $(V, |\cdot|_V)$ consisting of an A-module V and a map $|\cdot|_V : V \to \Gamma \cup \{0\}$ such that:

(i) $|x - y|_V \leq \max(|x|_V, |y|_V)$ for every $x, y \in V$.
(ii) $|ax| = |a| \cdot |x|_V$ for every $a \in A$ and $x \in V$.

If moreover $|\cdot|_V$ satisfies also the axiom:

(iii) $|x|_V = 0$ if and only if $x = 0$

then we say that $(V, |\cdot|_V)$ is a faithfully Γ-normed A-module. In the following we will suppose that all the Γ-seminormed A-modules under consideration are faithfully seminormed, so we shall refer to them simply as “seminormed A-modules” (or “normed A-modules” if (iii) holds).

1.2.1. Let $V := (V, |\cdot|_V)$ be a free seminormed A-module of finite rank. Following [6, §2.4.1, Def.1], we say that V is *A-cartesian* if there exists a basis $\{v_1, \dots, v_n\}$ of V such that

$$\left| \sum_{i=1}^n a_i v_i \right|_V = \max_{1 \leq i \leq n} |a_i| \cdot |v_i|_V$$

for all $a_1, \dots, a_n \in A$. A basis with this property is called *A-orthogonal* (or just orthogonal).

1.2.2. Suppose that $(A, |\cdot|)$ is a valuation ring. Recall that an *immediate extension* of A is a flat morphism of valuation rings $(A, |\cdot|) \to (A', |\cdot|')$ inducing an isomorphism of value groups $\Gamma \xrightarrow{\sim} \Gamma'$ and residue fields $A^\sim \xrightarrow{\sim} A'^\sim$. For instance, the henselization $(A^h, |\cdot|^h)$ of $(A, |\cdot|)$ is an immediate extension; also the completion $(A^\wedge, |\cdot|^\wedge)$ of $(A, |\cdot|)$ relative to its valuation topology, is an immediate extension.

Lemma 1.2.3. *Let $(A, |\cdot|)$ be a valuation ring, $(A', |\cdot|')$ an immediate extension of A, and B a flat A-algebra; set $B' := A^\wedge \otimes_A B$. We endow B (resp. B') with the spectral seminorm $|\cdot|_{\mathrm{sp}}$ (resp. $|\cdot|'_{\mathrm{sp}}$) relative to the valuation of A (resp. of A'). Suppose furthermore that both B and B' are reduced. Then:*

(i) *If $(B, |\cdot|_{\mathrm{sp}})$ is A-cartesian, then $(B', |\cdot|'_{\mathrm{sp}})$ is A'-cartesian. More precisely, a subset $\{b_1, \dots, b_d\}$ is an orthogonal basis of B if and only if $\{1 \otimes b_1, \dots, 1 \otimes b_d\}$ is an orthogonal basis of B'.*
(ii) *Conversely, suppose that $(B', |\cdot|'_{\mathrm{sp}})$ is A'-cartesian, and assume that $(A', |\cdot|') \subset (A^\wedge, |\cdot|^\wedge)$, the completion of A for the valuation topology. Then $(B, |\cdot|_{\mathrm{sp}})$ is A-cartesian.*

Proof. Notice that B is free of finite rank over A if and only if $B^\wedge$ is free of finite rank over $A^\wedge$ ([25, Rem.3.2.26(ii)]), hence we may assume from start that B is free of finite rank.

(i): Suppose that $\{b_1,\dots,b_d\}$ is an orthogonal basis of B; for every $a'_1,\dots,a'_d\in A^\wedge$ such that $x:=\sum_{i=1}^d a'_i\otimes b_i\neq 0$, we have $|x|'_{\mathrm{sp}}\neq 0$, since by assumption $B^\wedge$ is reduced. Since A' is an immediate extension of A, we can find $a_1,\dots,a_d\in A$ such that

$$\text{(1.2.4)}\qquad \text{either}\quad a_i=0 \qquad \text{or} \qquad |a_i-a'_i|<|a'_i| \qquad \text{for every } i\leq d$$

and especially, $|a_i|=|a'_i|$ for every $i\leq d$. Set $y:=\sum_{i=1}^d a_ib_i$; we deduce : $|x-1\otimes y|'_{\mathrm{sp}}<\max(|a_i|\cdot|b_i|_{\mathrm{sp}})=|y|_{\mathrm{sp}}=|1\otimes y|'_{\mathrm{sp}}$, by lemma 1.1.17(i.a), whence :

$$|x|'_{\mathrm{sp}}=|y|_{\mathrm{sp}}=\max(|a'_i|\cdot|1\otimes b_i|'_{\mathrm{sp}}\mid i=1,\dots,d)$$

i.e. $\{1\otimes b_1,\dots,1\otimes b_d\}$ is an orthogonal basis. Conversely, if $\{1\otimes b_1,\dots,1\otimes b_d\}$ is orthogonal, obviously $\{b_1,\dots,b_d\}$ is orthogonal in B.

(ii): In view of (i), we can assume that $B^\wedge$ is $A^\wedge$-cartesian, and it remains to show that B is A-cartesian. Choose an orthogonal basis $e'_1,\dots,e'_d$ for $B^\wedge$. We shall use the following analogue of (1.2.4) :

Claim 1.2.5. We can find $e_1,\dots,e_n\in B$ such that $|e_i-e'_i|'_{\mathrm{sp}}<|e'_i|'_{\mathrm{sp}}$ for $i=1,\dots,d$.

Proof of the claim. Write $e'_i=\sum_{j=1}^d a'_j\otimes b_j$ for some $a'_1,\dots,a'_d\in A^\wedge$ and $b_1,\dots,b_d\in B$; choose approximations $a_1,\dots,a_d\in A^\wedge$ of these elements and set $e_i:=\sum_{j=1}^d a_jb_j$. By lemma 1.1.17(i.b) we have : $|e_i-e'_i|_{\mathrm{sp}}\leq\max(|a_j-a'_j|\cdot|b_j|_{\mathrm{sp}}\mid j\leq d)$, which can be made arbitrarily small. ◊

It follows that $|e_i|_{\mathrm{sp}}=|e'_i|'_{\mathrm{sp}}$ and $(e_i\mid i=1,\dots,d)$ is a basis of B (by Nakayama's lemma); furthermore, for every $a_1,\dots,a_d\in A$ we have:

$$\begin{aligned}|\sum_{i=1}^d a_ie_i-\sum_{i=1}^d a_ie'_i|'_{\mathrm{sp}}=|\sum_{i=1}^d a_i(e_i-e'_i)|'_{\mathrm{sp}}&\leq\max(|a_i|\cdot|e_i-e'_i|'_{\mathrm{sp}}\mid i=1,\dots,d)\\ &<\max(|a_i|\cdot|e'_i|'_{\mathrm{sp}}\mid i=1,\dots,d)=|\sum_{i=1}^d a_ie'_i|'_{\mathrm{sp}}.\end{aligned}$$

Hence:

$$|\sum_{i=1}^d a_ie_i|_{\mathrm{sp}}=|\sum_{i=1}^d a_ie'_i|'_{\mathrm{sp}}=\max(|a_i|\cdot|e'_i|'_{\mathrm{sp}}\mid 1\leq i\leq d)=\max(|a_i|\cdot|e_i|_{\mathrm{sp}}\mid 1\leq i\leq d).$$

In other words, the basis $(e_i\mid i=1,\dots,d)$ is orthogonal. □

Remark 1.2.6. I do not know whether lemma 1.2.3(ii) holds for an arbitrary immediate extension $(A,|\cdot|)\to(A',|\cdot|')$. If the rank of the valuation ring A is greater than one, this seriously limits the usefulness of lemma 1.2.3, since for instance, for such valuations, the henselization A^h of A is not necessarily contained in $A^\wedge$.

1.2.7. Let $(K,|\cdot|)$ be a valued field with value group Γ and residue field $K^\sim$, L a finite extension of K and

$$|\cdot|_i:L\to\Gamma_i\cup\{0\}\qquad i=1,\dots,k$$

the finitely many extensions of $|\cdot|$. For every $i\leq k$, let $L_i^\sim$ be the residue field of the valuation ring L_i^+ of $(L,|\cdot|_i)$, and set $f_i:=[L_i^\sim:K^\sim]$, $e_i:=(\Gamma_i:\Gamma)$. Furthermore, for every pair of integers $i,j\leq k$ let Γ_{ij} be the value group of the valuation ring $L_{ij}^+:=L_i^+\cdot L_j^+$; the embedding $L_i\subset L_{ij}$ induces a natural surjective order-preserving group homomorphisms $\Gamma_i\to\Gamma_{ij}$, whose kernel we denote Δ_{ij}. Then we have natural isomorphisms of ordered groups $\Gamma_i/\Delta_{ij}\simeq\Gamma_j/\Delta_{ji}$, for every such pair (i,j). For every $i\leq k$ set also $\Delta_i:=\bigcap_{j\neq i}\Delta_{ij}$.

Proposition 1.2.8. *In the situation of* (1.2.7)*, endow the K-algebra L with its spectral norm $|\cdot|_{\mathrm{sp}}$ (relative to the norm $|\cdot|$ on K), and suppose moreover that :*

(a) *The extension $K\subset L$ is* defectless, i.e. $\sum_{i=1}^d e_i\cdot f_i=[L:K]$.

(b) $\Gamma+\Delta_i=\Gamma_i$ *for every* $i\leq k$.
(c) *For every* $i\leq k$, *the quotient* $\Delta_i/(\Gamma\cap\Delta_i)$ *consists of equivalence classes* $\overline{\alpha}_{i1},\dots,\overline{\alpha}_{ie_i}$ *of elements of* Δ_i *:*

$$\alpha_{i1}:=1>\alpha_{i2}>\cdots>\alpha_{ie_i}$$

such that $\alpha_{ij}>\gamma$ *for every* $i\leq k$, *every* $j\leq e_i$, *and every* $\gamma\in\Gamma^+\setminus\{1\}$.
(d) $\Delta_i\neq\{1\}$ *for every* $i\leq k$.

Then $(L,|\cdot|_{\mathrm{sp}})$ *is a cartesian* $(K,|\cdot|)$*-module.*

Proof. Assumption (c) and [44, Th.5] imply that for every $i\leq k$ and every $j\leq e_i$ we may find $x_{ij}\in L$ such that :

$$|x_{ij}|_i=\alpha_{ij}\qquad\text{and}\qquad|x_{ij}|_l=1\quad\text{for every } l\neq i. \tag{1.2.9}$$

Next, for every $i\leq k$, let $c_{i1},\dots,c_{if_i}\in L_i^+$ whose equivalence classes form a basis of the $K^\sim$-vector space $L_i^\sim$. In view of assumption (d) we may find, for every $i\leq k$, a non-zero element $b_i\in K$ such that :

$$|b_i|\in\Gamma^+\cap\Delta_i\setminus\{1\}.$$

Then, according to [44, Lemme 9] we may find, for every $i\leq k$ and every $j\leq f_i$, elements $y_{ij}\in L$ such that :

$$|y_{ij}-c_{ij}|_i\leq|b_i|\qquad\text{and}\qquad|y_{ij}|_l\leq|b_i|\quad\text{for every } l\neq i. \tag{1.2.10}$$

The proposition now follows from assumptions (a) and (b), and the following :

Claim 1.2.11. The family :

$$\Sigma:=(x_{ij}y_{il}\mid i=1,\dots,k;\ j=1,\dots,e_i;\ l=1,\dots,f_i)$$

is orthogonal relative to the spectral norm $|\cdot|_{\mathrm{sp}}$.

Proof of the claim. Let $a\in L$ be an element which can be written in the form :

$$a=\sum_{i=1}^k a_i\qquad\text{where :}\qquad a_i=\sum_{j=1}^{e_i}\sum_{l=1}^{f_i}a_{ijl}x_{ij}y_{il}\quad\text{for every } i=1,\dots,k$$

with $a_{ijl}\in K$. We have to show that :

$$|a|_{\mathrm{sp}}=\max(|a_{ijl}|\cdot|x_{ij}y_{il}|_{\mathrm{sp}}\mid i=1,\dots,k;\ j=1,\dots,e_i;\ l=1,\dots,f_i).$$

Let (K^h,L^h) be the henselization of $(K,|\cdot|)$, and set $L^{h+}:=L^+\otimes_{K^+}K^{h+}$, where $L^+:=\{x\in L\mid |x|_{\mathrm{sp}}\leq 1\}$ is the integral closure of K^+ in L. In view of lemma 1.1.17(i.a), it suffices to verify that Σ is an orthogonal system of elements of $(L^h,|\cdot|^h_{\mathrm{sp}})$, where $L^h=L^{h+}\otimes_{K^+}K$ and $|\cdot|_{\mathrm{sp}}$ is the spectral norm of L^h relative to $|\cdot|^h$. However, $L^h=L^h_i\times\cdots\times L^h_k$, where $(L^h_i,|\cdot|^h_i)$ denotes the henselization of $(L_i,|\cdot|_i)$, for every $i\leq k$, hence lemma 1.1.17(ii) yields the identity :

$$|a|_{\mathrm{sp}}=\max(|a|_i\mid i=1,\dots,k). \tag{1.2.12}$$

For every $i\leq k$ let us set :

$$\gamma_i:=\max(|a_{ijl}|\mid j=1,\dots,e_i;\ l=1,\dots,f_i).$$

In view of lemma 1.1.17(i.b) we may assume that :

$$\max(\gamma_i\mid i=1,\dots,k)=1.$$

Fix $i\leq k$. By inspecting (1.2.9) and (1.2.10) we see that in that case :

$$|a_t|_i\leq|b_i|\qquad\text{for every } t\neq i.$$

whence :

(1.2.13) $$|a|_i \leq \max(|a_i|_i, |b_i|) \qquad \text{and equality holds if } |a_i|_i > |b_i|.$$

• Now, suppose first that $\gamma_i = 1$, and denote by j_0 the smallest integer $j \leq e_i$ such that $|a_{ijl}| = 1$ for some $l \leq f_i$. With this notation, we may decompose :

$$a_i = a'_i + x_{ij_0} \cdot \sum_{l=1}^{f_i} a_{ij_0l} y_{il}$$

where a'_i is the sum of the terms $a_{ijl}x_{ij}y_{il}$ with $j \neq j_0$. By (1.2.10), the residue classes in $L_i^\sim$ of the elements $y_{i1}, \dots, y_{if_i}$ are all distinct, hence (1.2.9) yields :

$$|x_{ij_0} \cdot \sum_{l=1}^{f_i} a_{ij_0l} y_{il}|_i = \alpha_{ij_0}$$

and, on the other hand :

$$|a'_i|_i \leq \alpha_{i,j_0+1}$$

since α_{i,j_0+1} is greater than any element in $\Gamma^+ \setminus \{1\}$. Summing up we obtain :

(1.2.14) $$|a_i|_i = \alpha_{ij_0} = \max(|a_{ijl}| \cdot |x_{ij}y_{il}|_{\mathrm{sp}} \mid j = 1, \dots, e_i;\ l = 1, \dots, f_i) \qquad \text{if } \gamma_i = 1$$

and combining with (1.2.13) :

(1.2.15) $$|a|_i = \alpha_{ij_0} \qquad \text{whenever } \gamma_i = 1.$$

• Suppose next that $\gamma_i < 1$. Then $|a_i|_i \leq \gamma_i$, and in view of (1.2.13) we deduce :

(1.2.16) $$|a|_i \leq \max(|\gamma_i|, |b_i|) \qquad \text{whenever } \gamma_i < 1.$$

Finally, from (1.2.12), (1.2.15) and (1.2.16) we conclude that, in order to evaluate $|a|_{\mathrm{sp}}$ we may neglect all the terms a_{ijl} such that $\gamma_i < 1$, and then the sought identity follows from (1.2.14). □

Remark 1.2.17. Of the four conditions of proposition 1.2.8, the first one is very natural, and in fact characterizes cartesian extensions of a valued fields, in the rank one case (see [6, §3.6.2, Prop.5]). On the other hand, conditions (b), (c) and (d) appear (to me) as artifical, and certainly leave room for improvements.

1.2.18. Suppose that $V := (V, |\cdot|_V) \neq 0$ and $W := (W, |\cdot|_W)$ are two normed A-modules; let $\psi : V \to W$ be an A-linear homomorphism. We say that ψ is *bounded* if there exists $\gamma \in \Gamma$ such that

$$|\psi(x)|_W / |x|_V \leq \gamma \qquad \text{for every } x \in V \setminus \{0\}.$$

We denote by $\mathscr{L}(V, W)$ the A-module of all bounded A-linear homomorphisms $V \to W$. If $\Gamma \subset \mathbb{R}$, then one can define the norm of ψ as the supremum of $|\psi(x)|_W / |x|_V$ for x ranging over all $x \in V \setminus \{0\}$ ([6, §2.1.6]). For more general groups Γ, this quantity is not necessarily defined. Hence, for $\psi \in \mathscr{L}(V, W)$ we shall set

$$|\psi|_{\mathscr{L}} := \sup_{x \in V \setminus \{0\}} \frac{|\psi(x)|_W}{|x|_V}$$

whenever this is well defined as an element of $\Gamma \cup \{0\}$. Lemma 1.2.19 shows that, if V and W are A-cartesian, the norm $|\psi|_{\mathscr{L}}$ is well defined for every A-linear map ψ.

Lemma 1.2.19. *Let $V := (V, |\cdot|_V)$ and $W := (W, |\cdot|_W)$ be two free A-cartesian normed A-modules of finite rank. Then:*

(i) *$\mathscr{L}(V, W) = \mathrm{Hom}_k(V, W)$ and the pair $(\mathscr{L}(V, W), |\cdot|_{\mathscr{L}})$ is an A-cartesian normed A-module.*

(ii) *If* $(v_i \mid i = 1, \dots, n)$ *and* $(w_j \mid j = 1, \dots, m)$ *are orthogonal basis of* V*, resp.* W*, then the basis* $(v_i^* \otimes w_j \mid i = 1, \dots, n; j = 1, \dots, m)$ *of* $\mathscr{L}(V, W)$ *is orthogonal.*

Proof. (i) will follow from the more precise assertion (ii). The basis in (ii) is characterized by the identities

$$v_i^* \otimes w_j(v_k) = \delta_{ik} \cdot w_j \qquad \text{for every } i = 1, \dots, n \text{ and } j = 1, \dots, m.$$

Claim 1.2.20. $|v_i^* \otimes w_j|_{\mathscr{L}} = |w_j|_W / |v_i|_V$.

Proof of the claim. By definition we have

$$|v_i^* \otimes w_j|_{\mathscr{L}} = \sup_{\underline{b} \in A^n \setminus \{0\}} \frac{|b_i| \cdot |w_j|_W}{\max\limits_{1 \leq k \leq n} |b_k| \cdot |v_k|_V}.$$

For given $\underline{b} := (b_1, \dots, b_n)$, in order for the expression on the right-hand side to be non-zero, it is necessary that $b_i \neq 0$; in that case the denominator of the right-hand side cannot be made lower than $|b_i| \cdot |v_i|_V$, so the claim follows. $\Diamond$

Taking into account claim 1.2.20, the lemma boils down to the following

Claim 1.2.21. For every $n \times m$ matrix (α_{ij}) with coefficients in A we have:

$$\sup_{\underline{b} \in A^n \setminus \{0\}} \frac{\max\limits_{1 \leq j \leq m} |\sum_i \alpha_{ij} b_i| \cdot |w_j|_W}{\max\limits_{1 \leq k \leq n} |b_k| \cdot |v_k|_V} = \max_{ij} |\alpha_{ij}| \cdot \frac{|w_j|_W}{|v_i|_V}.$$

Proof of the claim. The inequality $\geq$ can be shown by choosing, for every $r \leq n$, the vector $\underline{b}_r := (b_{1r}, \dots, b_{nr})$ such that $b_{ir} = 0$ for $i \neq r$ and $b_{rr} = 1$. For the inequality $\leq$ one remarks that

$$\frac{\max\limits_{1 \leq j \leq m} |\sum_i \alpha_{ij} b_i| \cdot |w_j|_W}{\max\limits_{1 \leq k \leq n} |b_k| \cdot |v_k|_V} \leq \frac{\max\limits_{ij} |\alpha_{ij} b_i| \cdot |w_j|_W}{\max\limits_{1 \leq k \leq n} |b_k| \cdot |v_k|_V} \leq \max_{ij} \frac{|\alpha_{ij} b_i| \cdot |w_j|_W}{|b_i| \cdot |v_i|_V}$$

from which the claim follows easily. $\Box$

1.2.22. Let V and W be as in lemma 1.2.19. By lemma 1.2.19(i) there is a natural A-linear isomorphism

$$\mathscr{L}(V \otimes_A W, A) \simeq \mathscr{L}(V, \mathscr{L}(W, A))$$

whence a natural structure of A-cartesian normed A-module on $\mathscr{L}(V \otimes_k W, A)$. After dualizing (and applying again lemma 1.2.19) we deduce that $V \otimes_A W$ carries a natural structure of A-cartesian normed A-module. Furthermore, let $(v_i \mid i = 1, \dots, n)$ and $(w_j \mid j = 1, \dots, m)$ be orthogonal bases for V and respectively W; using repeatedly lemma 1.2.19(ii) one sees easily that $(v_i \otimes w_j \mid i = 1, \dots, n; j = 1, \dots, m)$ is an orthogonal basis for $V \otimes_A W$ and moreover

$$|v_i \otimes w_j| = |v_i|_V \cdot |w_j|_W \qquad \text{for every } i = 1, \dots, n \text{ and } j = 1, \dots, m.$$

Remark 1.2.23. At least when $\Gamma = \mathbb{R}$, and A is a field, it should be possible to use the characterization of [6, §2.1.7, Cor.3] to see that the above normed A-module structure on $V \otimes_A W$ agrees with the one defined on the complete tensor product $V \widehat{\otimes}_A W$ as in [6, §2.1.7].

1.2.24. As a special case of (1.2.22), we deduce a natural norm on every tensor power $V^{\otimes k}$ of V. All these A-modules are A-cartesian. For every $k \in \mathbb{N}$ we have a natural imbedding of A-modules: $\Lambda^k_A V \hookrightarrow V^{\otimes k}$ induced by the antisymmetrizer operator ([8, Ch.III, §7.4, Remarque])

$$V^{\otimes k} \to V^{\otimes k} \quad : \quad v_1 \otimes \cdots \otimes v_k \mapsto \sum_{\sigma \in S_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

Hence the norm of $V^{\otimes k}$ restricts to a natural norm on $\Lambda^k_A V$. Let $\{v_1, \dots, v_n\}$ be an orthogonal basis for V. For every subset $I \subset \{1, \dots, n\}$ of cardinality $|I| = k$ we set $v_I := v_{i_1} \wedge \cdots \wedge v_{i_k}$, where $i_1 < \cdots < i_k$ are the elements of I. One checks easily that

$$|v_I| = |v_{i_1}|_V \cdot \ldots \cdot |v_{i_k}|_V$$

and the basis $(v_I \mid I \subset \{1, \dots, n\}, |I| = k)$ is orthogonal.

1.2.25. In the situation of (1.2.24), consider a free A^+-submodule $V^+ \subset V$ such that the natural map $A \otimes_{A^+} V^+ \to V$ is an isomorphism. The highest exterior power $\Lambda^n_{A^+} V^+$ is a rank one free A^+-submodule of the A-cartesian module $\Lambda^n_A V$. Pick any generator e of $\Lambda^n_{A^+} V^+$; one sees easily that the value

$$|V^+| := |e|$$

is independent of the choice of e. Especially, if A is an integral domain and $I \subset A^+$ is any principal ideal, then $|I|$ is well defined.

Lemma 1.2.26. *Suppose that A is an integral domain and let $V_1^+ \subset V_2^+ \subset V$ be two A^+-submodules of the free cartesian A-module V of finite rank, fulfilling the conditions of* (1.2.25)*. Then we have:*

$$|V_1^+| = |F_0(V_2^+/V_1^+)| \cdot |V_2^+|.$$

Proof. Here F_0 denotes the Fitting ideal (see [36, Ch.XIX] for generalities on Fitting ideals). Let n be the rank of V; more or less by definition we have $F_0(V_2^+/V_1^+) = F_0(\Lambda^n_{A^+} V_2^+/\Lambda^n_{A^+} V_1^+)$, from which the assertion follows easily. □

1.3. **Henselian algebras and complete algebras.** Let $(K, |\cdot|)$ be a complete valued field of rank one, $\mathfrak{m}$ the maximal ideal of the valuation ring K^+ of K, $K^\sim := K^+/\mathfrak{m}$ the residue field and Γ_K the value group. Let also $\pi \in \mathfrak{m}$ be a fixed non-zero element.

1.3.1. For any K^+-algebra R, let us denote by $R\text{-}\mathbf{Alg}_{\text{fpét}/K}$ (resp. $R\text{-}\mathbf{Alg}_{\text{fgét}/K}$) the category of R-algebras B that are finitely presented (resp. finitely generated) as R-modules and such that $B_K := B \otimes_{K^+} K$ is étale over $R_K := R \otimes_{K^+} K$. Furthermore, for any object B of $R\text{-}\mathbf{Alg}_{\text{fgét}/K}$, let B^ν be the integral closure in B_K of the image of B.

Proposition 1.3.2. *With the notation of* (1.3.1)*, suppose that R is K^+-flat and henselian along its ideal πR, and denote by $R^\wedge$ the π-adic completion of R. Then :*

(i) *The base change functor $B \mapsto B^\wedge := R^\wedge \otimes_R B$ induces equivalences :*

$$R\text{-}\mathbf{Alg}_{\text{fpét}/K} \xrightarrow{\sim} R^\wedge\text{-}\mathbf{Alg}_{\text{fpét}/K} \qquad R\text{-}\mathbf{Alg}_{\text{fgét}/K} \xrightarrow{\sim} R^\wedge\text{-}\mathbf{Alg}_{\text{fgét}/K}.$$

(ii) $\mathrm{Ann}_{B^\wedge}(\pi) = \mathrm{Ann}_B(\pi)$ *for every object B of $R\text{-}\mathbf{Alg}_{\text{fgét}/K}$.*

(iii) *Suppose furthermore that R_K and $R^\wedge_K$ are normal domains. Then the natural map :*

$$B^\nu \otimes_R R^\wedge \to (B \otimes_R R^\wedge)^\nu$$

is an isomorphism for every object B of $R\text{-}\mathbf{Alg}_{\text{fgét}/K}$.

Proof. (i): The assertion for R-$\mathbf{Alg}_{\mathrm{fpét}/K}$ follows from [23, Ch.III, §3, Th.5 and §4, Rem.2, p.587]. Next, let $B^\wedge$ be any object of $R^\wedge$-$\mathbf{Alg}_{\mathrm{fgét}/K}$; we may find a filtered system $(B^\wedge_\lambda \mid \lambda \in \Lambda)$ of finitely presented $R^\wedge$-algebras, with surjective transition maps $\phi^\wedge_{\lambda\mu} : B^\wedge_\lambda \to B^\wedge_\mu$, whose colimit is B. Since $B^\wedge$ is integral over $R^\wedge$, we may also arrange that $B^\wedge_\lambda$ is integral over $R^\wedge$ for every $\lambda \in \Lambda$, and then every $B^\wedge_\lambda$ is of finite presentation as $R^\wedge$-module. Furthermore, since $B^\wedge_K$ is a finitely presented $R^\wedge_K$-algebra, we may assume that

$$B^\wedge_\lambda \otimes_{K^+} K \simeq B^\wedge \qquad \text{for every } \lambda \in \Lambda. \tag{1.3.3}$$

Therefore every $B^\wedge_\lambda$ is an object of $R^\wedge$-$\mathbf{Alg}_{\mathrm{fpét}/K}$. In this case, under the foregoing equivalence, this family comes from a filtered system $(B_\lambda \mid \lambda \in \Lambda)$ of objects of R-$\mathbf{Alg}_{\mathrm{fpét}/K}$. Let B be the colimit of the latter family; then $B \otimes_R R^\wedge \simeq B^\wedge$. Moreover :

Claim 1.3.4. (i) B is a finitely generated R-module.

(ii) The induced map : $B_\lambda \otimes_{K^+} K \to B_K$ is bijective for every $\lambda \in \Lambda$.

Proof of the claim. For (i), it suffices to show that the transition maps $\phi_{\lambda\mu} : B_\lambda \to B_\mu$ are still surjective. Indeed, let $C_{\lambda\mu} := \operatorname{Coker} \phi_{\lambda\mu}$; then $C_{\lambda\mu} \otimes_R R^\wedge = 0$, hence $C_{\lambda\mu}/\pi C_{\lambda\mu} = 0$, and therefore $C_{\lambda\mu} = 0$, by Nakayama's lemma.

(ii): Let $C_\lambda := B/B_\lambda$; we have to show that $\pi^n C_\lambda = 0$ for large enough $n \in \mathbb{N}$. However, $C_\lambda/\pi^n C_\lambda \simeq C_\lambda \otimes_R (R/\pi^n R) \simeq C_\lambda \otimes_R (R^\wedge/\pi^n R^\wedge) \simeq C_\lambda \otimes_R R^\wedge$ for every sufficiently large $n \in \mathbb{N}$, by (1.3.3). It follows that $C_\lambda/\pi^n C_\lambda = C_\lambda/\pi^m C_\lambda$ for every $m > n$, *i.e.* $\pi^n C_\lambda = \pi^m C_\lambda$, whence $\pi^m C_\lambda = 0$ by (i) and Nakayama's lemma. ◇

Claim 1.3.4 implies that B is an object of R-$\mathbf{Alg}_{\mathrm{fgét}/K}$, hence the base change functor is essentially surjective on the subcategory $R^\wedge$-$\mathbf{Alg}_{\mathrm{fgét}/K}$. Next, let C be any other object of R-$\mathbf{Alg}_{\mathrm{fgét}/K}$, and $\alpha^\wedge : B^\wedge \to C^\wedge$ a ring homomorphism. Choose a filtered system $(C_\mu \mid \mu \in \Lambda')$ of objects of R-$\mathbf{Alg}_{\mathrm{fpét}/K}$ whose colimit is C; for every $\lambda \in \Lambda$, let $\psi_\lambda : B^\wedge_\lambda \to B^\wedge$ be the natural map; we may find $\mu \in \Lambda'$ such that the composition $\alpha^\wedge \circ \phi_\lambda$ factors though a morphism $\alpha^\wedge_{\mu\lambda} : B^\wedge_\lambda \to C^\wedge_\mu$. Let $\alpha_{\mu\lambda} : B_\lambda \to C_\mu$ be the corresponding morphism in R-$\mathbf{Alg}_{\mathrm{fpét}/K}$ and $\alpha_\lambda : B_\lambda \to C$ its composition with the natural map $C_\mu \to C$. One checks easily that α_λ does not depend on the choice of $\alpha_{\mu\lambda}$, and moreover, the collection $(\alpha_\lambda \mid \lambda \in \Lambda)$ is compatible with the transition morphisms $\phi_{\lambda\lambda'} : B_\lambda \to B_{\lambda'}$ of the filtered system $(B_\lambda \mid \lambda \in \Lambda)$, therefore it gives rise to a map $\alpha : B \to C$, and by construction it is clear that $\alpha \otimes_R \mathbf{1}_{R^\wedge} = \alpha^\wedge$. This shows that the base change is a full functor R-$\mathbf{Alg}_{\mathrm{fgét}/K} \to R^\wedge$-$\mathbf{Alg}_{\mathrm{fgét}/K}$. A similar argument, by reduction to finitely presented algebras, yields also the faithfulness of the functor, thus concluding the proof of assertion (i).

(ii): To start out, we claim that the natural map $B \to B^\wedge$ is injective for every object B of R-$\mathbf{Alg}_{\mathrm{fgét}/K}$. Indeed, let $I \subset B$ be the kernel of this map; then both $B^\wedge$ and $(B/I)^\wedge$ are the coproduct of B and $R^\wedge$ in the category of R-algebras, hence the natural map $B^\wedge \to (B/I)^\wedge$ is an isomorphism, so the same holds for the map $B \to B/I$, in view of (i). Now, let $T := \operatorname{Ann}_B(\pi)$; clearly $T \otimes_R R^\wedge = T$, whence a right exact sequence : $T \to B^\wedge \xrightarrow{\pi} B^\wedge \to 0$. However, T injects into $B^\wedge$, since it injects into B and B injects into $B^\wedge$; hence the foregoing sequence is short exact, which is assertion (ii).

(iii): Under the standing assumptions, B^ν is the colimit of the filtered family $(B_\lambda \mid \lambda \in \Lambda)$ consisting of all the objects of R-$\mathbf{Alg}_{\mathrm{fgét}/K}$ such that $B_\lambda \otimes_{K^+} K = B$ and such that π is regular in B_λ. In view of (ii), the family $(B^\wedge_\lambda \mid \lambda \in \Lambda)$ consists of all the objects of $R^\wedge$-$\mathbf{Alg}_{\mathrm{fgét}/K}$ such that $B^\wedge_\lambda \otimes_{K^+} K = B$ and such that π is regular in $B^\wedge_\lambda$, hence its colimit is $(B^\wedge)^\nu$. □

1.3.5. Let A a K^+-algebra of finite presentation, and A^h (resp. $A^\wedge$) the henselization of A along its ideal πA (resp. the π-adic completion of A). Recall that the π-adic completion of A^h is naturally isomorphic to $A^\wedge$. (indeed, $A^h/\pi^n A^h$ is the henselization of $A/\pi^n A$ along its

ideal $\pi A/\pi^n A$, for every $n \in \mathbb{N}$ [43, Ch.XI, §2, Prop.2]; therefore the natural map $A/\pi^n A \to A^h/\pi^n A^h$ is an isomorphism for every $n \in \mathbb{N}$, which implies the claim).

Lemma 1.3.6. *In the situation of* (1.3.5)*, suppose that A is flat over K^+. Then $A^\wedge$ is faithfully flat over A^h.*

Proof. To begin with, we claim that $A^\wedge$ is flat over K^+. Indeed, suppose that $\pi x = 0$ for some $x \in A^\wedge$; choose a sequence $(x_k \mid k \in \mathbb{N})$ of elements of A converging to x in the π-adic topology of $A^\wedge$. Then the sequence $(\pi x_k \mid k \in \mathbb{N})$ converges to 0, and since A has no π-torsion, it follows easily that the sequence $(x_k \mid k \in \mathbb{N})$ also converges to 0, so $x = 0$.

Claim 1.3.7. In order to show the lemma, it suffices to prove that $A^\wedge$ is flat over A.

Proof of the claim. First, if $A^\wedge$ is flat over A, then $A^\wedge$ is flat over A^h as well. To conclude, by standard reductions, it suffices to show that a finitely generated A^h-module M vanishes if and only if $M^\wedge := A^\wedge \otimes_{A^h} M$ vanishes. But if $M^\wedge = 0$, it follows that $M/\pi M = 0$, and then we invoke Nakayama's lemma to see that $M = 0$. ◊

Hence, let us that show $A^\wedge$ is flat over A; to this aim, since $A/\pi A \simeq A^\wedge/\pi A^\wedge$, [25, Lemma 5.2.1] says that it suffices to show that $A^\wedge_K := A^\wedge \otimes_{K^+} K$ is flat over $A_K := A \otimes_{K^+} K$. Say that $A = K^+[T_1, \dots, T_n]/I$ for some finitely generated ideal I.

Claim 1.3.8. $\pi^n I = I \cap \pi^n K^+[T_1, \dots, T_n]$ for every $n \in \mathbb{N}$.

Proof of the claim. By assumption $\mathrm{Tor}_1^{K^+}(A, K^+/\pi^n K^+) = 0$; hence

$$I/\pi^n I = \mathrm{Ker}(K^+[T_1, \dots, T_n]/\pi^n K^+[T_1, \dots, T_n] \to A/\pi^n A).$$

The claim follows easily. ◊

By [41, Th.8.4] we have $A^\wedge \simeq K^+\langle T_1, \dots, T_n\rangle/I^\wedge$, where $I^\wedge$ is the completion of I for the subspace topology as a submodule of $K^+[T_1, \dots, T_n]$; by claim 1.3.8 the subspace topology is nothing else than the π-adic topology. It then follows from [25, Prop.7.1.1](iv) that :

$$I^\wedge = IK^+\langle T_1, \dots, T_n\rangle$$

hence $A^\wedge \simeq K^+\langle T_1, \dots, T_n\rangle \otimes_{K^+[T_1,\dots,T_n]} A$ and $A^\wedge_K \simeq K\langle T_1, \dots, T_n\rangle \otimes_{K[T_1,\dots,T_n]} A_K$, hence we are reduced to the case where $A = K^+[T_1, \dots, T_n]$. Let $\mathfrak{n} \subset A^\wedge_K$ be any maximal ideal, and set $\mathfrak{q} := \mathfrak{n} \cap A_K$; it suffices to show that $A^\wedge_{K,\mathfrak{n}}$ is flat over $A_{K,\mathfrak{q}}$. However, it is well known that $E := A^\wedge_K/\mathfrak{n}$ is a finite extension of K, hence the same holds for $A_K/\mathfrak{q} \subset E$. Choose any maximal ideal $\mathfrak{n}_E \subset A^\wedge_E := E \otimes_K A^\wedge_K \simeq E\langle T_1, \dots, T_n\rangle$ lying over $\mathfrak{n}$ and let $\mathfrak{q}_E$ be the preimage of $\mathfrak{n}_E$ in $A_E := E[T_1, \dots, T_n]$. Since the extension $A^\wedge_{K,\mathfrak{n}} \to A^\wedge_{E,\mathfrak{n}_E}$ is faithfully flat, we are reduced to showing that $A^\wedge_{E,\mathfrak{n}_E}$ is flat over $A_{E,\mathfrak{q}_E}$. Hence we can replace K by E and assume from start that $A/\mathfrak{n} \simeq K$, in which case $\mathfrak{n} = (T_1 - a_1, \dots, T_n - a_n)$ for some $a_1, \dots, a_n \in K^+$. Clearly the $\mathfrak{n}$-adic completions of A_K and $A^\wedge_K$ are both isomorphic to $K[[T_1 - a_1, \dots, T_n - a_n]]$, and by [41, Th.8.8] this latter ring is faithfully flat over both $A_{K,\mathfrak{q}}$ and $A^\wedge_{K,\mathfrak{n}}$. The claim follows easily. □

2. Study of the discriminant

2.1. **Discriminant.** Let $R \to S$ be a ring homomorphism such that S is a free R-module of finite rank. Every element $a \in S$ defines an R-linear endomorphism

$$\mu_a : S \to S \qquad b \mapsto ab$$

whose trace and determinant we denote respectively by $\mathrm{tr}_{S/R}(a)$ and $\mathrm{Nm}_{S/R}(a)$. There follows a well-defined R-bilinear *trace form*

$$\mathrm{Tr}_{S/R} : S \otimes_R S \to R \qquad a \otimes b \mapsto \mathrm{tr}_{S/R}(ab).$$

It is well known (see *e.g.* [25, Th.4.1.14]) that $\mathrm{Tr}_{S/R}$ is a perfect pairing if and only if S is étale over R. Pick a basis $e_1, \dots, e_d$ of S; one defines the *discriminant* of S over R as the element

$$\mathfrak{d}_{S/R} := \det(\mathrm{Tr}_{S/R}(e_i \otimes e_j) \mid 1 \le i, j \le d) \in R.$$

One verifies easily that $\mathfrak{d}_{S/R}$ is well defined (*i.e.* independent of the choice of the basis) up to the square of an invertible element of R.

2.1.1. Let R be a (not necessarily commutative) unitary ring; for any integer $m > 0$ we let $M_m(R)$ be the unitary ring of all $m \times m$ matrices with entries in R. For every $a \in R$ and every pair of integers $i, j \le m$ we denote by $E_{ij}(a) \in M_m(R)$ the elementary matrix whose (i, j)-entry equals a, and whose other entries vanish; moreover, sometimes we may denote by 1_m the unit of $M_m(R)$. If $n > 0$ is any other integer, we let

$$\alpha_n : M_n(M_m(R)) \xrightarrow{\sim} M_{nm}(R)$$

be the unique ring isomorphism such that

$$E_{ij}(E_{kl}(a)) \mapsto E_{i(m-1)+k, j(m-1)+l}(a) \qquad \text{for all } a \in R, \quad 1 \le i, j \le n, \quad 1 \le k, l \le m.$$

Suppose now that $t := (t_{ij}) \in M_n(M_m(R))$ is a matrix whose entries t_{ij} commute pairwise; let $T \subset M_m(R)$ be the commutative ring generated by all the t_{ij}; we can then view t as an element of $M_n(T)$ so that its determinant is well-defined as an element of T. To avoid ambiguities, we shall write $\det_n(t_{ij} \mid 1 \le i, j \le n)$ for this determinant.

Lemma 2.1.2. *With the notation of* (2.1.1)*, suppose that the ring R is commutative and let $t := (t_{ij} \mid 1 \le i, j \le n) \in M_n(M_m(R))$ be an element such that all the matrices $t_{ij} \in M_m(R)$ commute pairwise. We have the identity:*

$$\det(\det_n(t_{ij} \mid 1 \le i, j \le n)) = \det(\alpha_n(t)). \tag{2.1.3}$$

Proof. We proceed by induction on n, the case $n = 1$ being trivial. Hence, assume $n > 1$; suppose first that $t_{11} \in M_m(R)$ is an invertible matrix. It follows that the matrix

$$s := E_{11}(t_{11}) + \sum_{k=2}^{n} E_{kk}(1_m)$$

is invertible in $M_n(M_m(R))$, and obviously its entries commute pairwise and with the entries of t; furthermore the sought identity is easily verified for s. Since both sides of (2.1.3) are multiplicative in t, we are therefore reduced to verifying the identity for $s^{-1} \cdot t$, hence we can assume that $t_{11} = 1_m$. Next, let

$$e := (1_m)_n - \sum_{k=2}^{n} E_{1k}(t_{1k}).$$

Clearly e is invertible in $M_n(M_m(R))$, and again its entries commute both pairwise and with the entries of t; thus it suffices to show the sought identity for e and for $e^{-1} \cdot t$. The identity is obvious for e, therefore we can replace t by $e^{-1} \cdot t$ and assume that $t_{1j} = \delta_{1j} \cdot 1_m$ for every $j \le n$. In this case, $\det_n(t_{ij} \mid 1 \le i, j \le n)$ equals the determinant of the $(n-1) \times (n-1)$-minor t' obtained by omitting the first row and the first column of t, and $\det(\alpha_n(t)) = \det(\alpha_{n-1}(t'))$. By inductive assumption, the sought identity is already know for such a minor, so the proof is complete in case t_{11} is invertible. For a general t, we notice that $\det(t_{11} + \lambda 1_m)$ is a non-zero-divisor in the free polynomial R-algebra $R[\lambda]$ and consider the localization $S := R[\lambda, \det(t_{11} + \lambda 1_m)^{-1}]$. The assumptions of the lemma are verified by the matrix $t'' := t + E_{11}(\lambda 1_m) \in M_n(M_m(S))$ and moreover t''_{11} is invertible in $M_n(S)$, so (2.1.3) holds for t'', and actually yield an identity in the subring $R[\lambda]$ of S. After specializing the latter identity in $\lambda = 0$, we deduce that (2.1.3) holds for t as well. $\square$

Proposition 2.1.4. *Let $A \to B \to C$ be maps of commutative rings and suppose that B (resp. C) is a free A-module (resp. B-module) of finite rank. Let $r := \mathrm{rk}_B C$. Then we have:*

$$\mathfrak{d}_{C/A} = (\mathfrak{d}_{B/A})^r \cdot \mathrm{Nm}_{B/A}(\mathfrak{d}_{C/B}).$$

Proof. Let $d := \mathrm{rk}_A B$; pick bases $e_1, \dots, e_d \in B$ of the A-module B and $f_1, \dots, f_r \in C$ of the B-module C; clearly the system $(e_i f_j \mid i \leq d, j \leq r)$ is a basis for the free A-module C of rank dr. We let $T \in M_r(M_d(A))$ be the element whose (j, j')-entry is the matrix $T_{jj'}$ such that

$$(T_{jj'})_{ii'} := \mathrm{tr}_{C/A}(e_i f_j e_{i'} f_{j'}) \qquad \text{for every } 1 \leq i, i' \leq d.$$

In the notation of (2.1.1), we have $\mathfrak{d}_{C/A} = \det(\alpha_r(T))$. Moreover, let $M \in M_r(B)$ (resp. $N \in M_d(A)$) be the matrix such that $M_{jj'} := \mathrm{tr}_{C/B}(f_j f_{j'})$ (resp. such that $N_{ii'} := \mathrm{tr}_{B/A}(e_i e_{i'})$); by the transitivity of the trace, we can write

$$(T_{jj'})_{ii'} = \mathrm{tr}_{B/A}(e_i e_{i'} \cdot M_{jj'}). \tag{2.1.5}$$

Let $\mu : B \to M_d(A)$ be the unique ring homomorphism such that

$$be_i = \sum_{k=1}^{d} \mu(b)_{ki} e_k \qquad \text{for all } b \in B.$$

Especially: $e_{i'} M_{jj'} = \sum_{k=1}^{d} \mu(M_{jj'})_{ki'} e_k$ and consequenly:

$$\mathrm{tr}_{B/A}(e_i e_{i'} M_{jj'}) = \sum_{k=1}^{d} \mu(M_{jj'})_{ki'} \cdot \mathrm{tr}_{B/A}(e_k e_i) = \sum_{k=1}^{d} N_{ik} \cdot \mu(M_{jj'})_{ki'}. \tag{2.1.6}$$

Finally, let $\Delta(N), \mu(M) \in M_r(M_d(A))$ be the matrices such that

$$\Delta(N)_{jj'} := N \cdot \delta_{jj'} \qquad \mu(M)_{jj'} := \mu(M_{jj'}) \qquad \text{for all } 1 \leq j, j' \leq r.$$

Taking into account (2.1.5) and (2.1.6) we see that

$$T = \Delta(N) \cdot \mu(M)$$

whence, an application of lemma 2.1.2 delivers the sought identity. $\square$

2.1.7. In the situation of (2.1) we let

$$\tau_{S/R} : S \to S^* := \mathrm{Hom}_R(S, R)$$

be the map such that $\tau_{S/R}(b)(b') = \mathrm{Tr}_{S/R}(b \otimes b')$ for every $b, b' \in S$. Notice that S^* is an S-module with the natural scalar multiplication defined by the rule: $(b \cdot \phi)(b') := \phi(bb')$ for every $b, b' \in S$ and $\phi \in S^*$. With respect to this S-module structure, τ is S-linear; thus, we can define the *different ideal*

$$\mathscr{D}_{S/R} := \mathrm{Ann}_S(\mathrm{Coker}\, \tau_{S/R}) \subset S.$$

In this generality, not much can be said about the ideal $\mathscr{D}_{S/R}$. However, suppose furthermore that there is an isomorphism of S-modules $\omega : S^* \xrightarrow{\sim} S$; it follows easily that $\mathscr{D}_{S/R}$ is the principal ideal generated by $\delta := \omega \circ \tau(1)$. Denote by $\mathrm{Nm}_{S/R}(\mathscr{D}_{S/R}) \subset R$ the principal ideal generated by $\mathrm{Nm}_{S/R}(\delta)$.

Lemma 2.1.8. *Under the assumptions of* (2.1.7) *we have the identity:*

$$\mathrm{Nm}_{S/R}(\mathscr{D}_{S/R}) = \mathfrak{d}_{S/R}.$$

Proof. Indeed, let $r := \mathrm{rk}_R S$; directly from the definition we deduce that

$$\mathfrak{d}_{S/R} = \mathrm{Ann}_R(\mathrm{Coker}\, \Lambda^r_R \tau_{S/R}) = \mathrm{Ann}_R(\mathrm{Coker}\, \Lambda^r_R(\omega \circ \tau_{S/R})).$$

which implies straightforwardly the assertion. $\square$

Example 2.1.9. Suppose that R is a henselian valuation ring and S is the integral closure of R in a finite extension of the field of fractions of R; moreover suppose that S is a finitely presented R-module. Then actually S is a free R-module of finite rank. Notice that S is a valuation ring and an S-module is S-torsion-free if and only if it is R-torsion-free; in particular we see that S^* is a finitely presented S-torsion-free S-module, hence it is free over S (see *e.g.* [25, lemma 6.1.14]) and clearly $\mathrm{rk}_S S^* = 1$, so lemma 2.1.8 applies to the extension $R \subset S$. Moreover :

Lemma 2.1.10. *Keep the assumptions of example* 2.1.9 *and suppose moreover that the valuation* $|\cdot|_R$ *of* R *has rank one; let* $d := \mathrm{rk}_R S$. *Then:*

(i) *In case the valuation* $|\cdot|_R$ *is discrete,* $|\mathfrak{d}_{S/R}| \geq |\pi|_R^{d(1-1/e)} \cdot |d|^d$, *where* $\pi \in R$ *is a uniformizer of* R *and* e *is the ramification index of* S *over* R.
(ii) *In case the valuation* $|\cdot|_R$ *is not discrete,* $|\mathfrak{d}_{S/R}| \geq |d|^d$.

Proof. Obviously $\mathrm{tr}_{S/R}(d^{-1}) = 1$ in either case. If the valuation of R is discrete, we can write $\mathrm{tr}_{S/R}(\pi^{-1} \cdot d^{-1}) = \pi^{-1}$, hence $\pi^{-1} \cdot d^{-1} \notin \mathscr{D}_{S/R}^{-1}$ and consequently $\mathscr{D}_{S/R}^{-1} \subset \pi_S \cdot \pi^{-1} \cdot d^{-1} S$ (inclusion of fractional ideals, where π_S is a uniformizer for S). The bound then follows easily from lemma 2.1.8. In case the valuation is non-discrete, the same argument yields the weaker estimate: $\mathscr{D}_{S/R}^{-1} \subset x \cdot d^{-1} S$, for every x with $|x|_R < 1$, whence the sought inequality, again by lemma 2.1.8. $\square$

Lemma 2.1.11. *Let* R *be a valuation ring and* $S_2 \subset S_1$ *two* R*-algebras that are both free of the same finite rank as* R*-modules. Then we have:*

$$\mathfrak{d}_{S_2/R} = F_0(S_1/S_2)^2 \cdot \mathfrak{d}_{S_1/R}.$$

Proof. (Here F_0 denotes the Fitting ideal of the torsion R-module S_1/S_2.) This is a special case of [25, Lemma 7.5.4]. $\square$

The following example will play a key role in later sections.

Example 2.1.12. Let K, K^+ and π be as in (1.3). As usual, one lets $K^+\langle T_1, \dots, T_n\rangle$ be the π-adic completion of $K^+[T_1, \dots, T_n]$. We consider the (continuous) ring homomorphism

$$\psi : K^+\langle\xi\rangle \to K^+\langle S, T\rangle/(ST - \pi^2) \quad : \quad \xi \mapsto S + T.$$

Notice that $K^+\langle S, T\rangle/(ST - \pi^2)$ is generated over $K^+\langle\xi\rangle$ by the class of S, which satisfies the integral equation

$$S^2 - S\xi + \pi^2 = 0.$$

The matrix of the trace form for the morphism ψ, relative to the basis $(1, S)$ is:

$$\begin{pmatrix} 2 & \xi \\ \xi & \xi^2 - 2\pi^2 \end{pmatrix}.$$

Finally, the discriminant of ψ is $\mathfrak{d}_\psi := \xi^2 - 4\pi^2$.

2.2. **Finite ramified coverings of annuli.** We keep the notation and assumptions of (1.3), and we suppose additionally that K is algebraically closed.

2.2.1. We shall use rather freely the language of adic spaces of [29] and [30]. For a quick review of the main definitions, we refer also to [25, §7.2.15-27]. Recall that such an adic space is a datum of the form $(X, \mathscr{O}_X, \mathscr{O}_X^+)$, where $(X, \mathscr{O}_X)$ is a locally ringed space and $\mathscr{O}_X^+ \subset \mathscr{O}_X$ is a subsheaf of rings satisfying certain natural conditions (see *loc.cit.*); moreover such an adic space is obtained by gluing *affinoid* open subspaces that are *adic spectra* $\mathrm{Spa}\, A$ attached to certain pairs $A := (A^\triangleright, A^+)$ consisting of a ring and an integrally closed subring $A^+ \subset A^\triangleright$.

For every $x \in X$ we shall denote :

- $\kappa(x)$ the residue field of $\mathscr{O}_{X,x}$, which is a valued field whose valuation we denote $|\cdot|_x$.

- $(\kappa(x)^\wedge, |\cdot|^\wedge_x)$ (resp. $(\kappa(x)^h, |\cdot|^h_x)$) the completion for the valuation topology (resp. the henselization) of $(\kappa(x), |\cdot|_x)$.
- $(\kappa(x)^{\wedge h}, |\cdot|^{\wedge h}_x)$ the henselization of $(\kappa(x)^\wedge, |\cdot|^\wedge_x)$.

In agreement with (1.1.3), we shall write $\kappa(x)^+$ for the valuation ring of the valuation $|\cdot|_x$, and likewise we define $\kappa(x)^{\wedge+}$ and so on. For future reference we point out :

Lemma 2.2.2. *Let X be an analytic adic space, $x \in X$ any point and $y \in X$ a specialization of x. Then the induced map $\kappa(y)^\wedge \to \kappa(x)^\wedge$ is an isomorphism of complete topological fields.*

Proof. It follows directly from [30, Lemma 1.1.10(iii)]. □

Remark 2.2.3. Let $f : X \to Y$ be a finite map of analytic adic spaces, $x \in X$ any point and $y := f(x)$. Notice that the extension of valued fields $(\kappa(y), |\cdot|_y) \subset (\kappa(x), |\cdot|_x)$ is usually *not* algebraic, whereas the induced map on completions :

$$(\kappa(y)^\wedge, |\cdot|^\wedge_y) \to (\kappa(x)^\wedge, |\cdot|^\wedge_x)$$

is always a finite algebraic extension ([30, Lemma 1.5.2]), but the latter does not necessarily induce a finite map between the corresponding valuation rings (see (2.2.16)). That is why it is useful to take henselizations : the induced ring homomorphism $\kappa(y)^{\wedge h+} \to \kappa(x)^{\wedge h+}$ is finite.

2.2.4. Following R.Huber ([29], [30]), we call a *K-affinoid algebra* a pair $A := (A^\triangleright, A^+)$, where $A^\triangleright$ is a K-algebra of topologically finite type and A^+ is a subring of the ring A° of all power-bounded elements of $A^\triangleright$. We shall consider exclusively affinoid rings A of *topologically finite type over* K; for such A one has always $A^+ = A^\circ$. The subring A° is characterized by a topological – rather than metric – condition. Hence in principle the notation of this section may conflict with (1.1.3). However, when $A^\triangleright$ is reduced, one knows that the *supremum seminorm* $|\cdot|_{\mathrm{sup}}$ on $A^\triangleright$ is a power-multiplicative norm ([6, §6.2.4, Th.1]), and furthermore in this case $A^\circ = \{a \in A^\triangleright \mid |a|_{\mathrm{sup}} \leq 1\}$ ([6, §6.2.3, Prop.1]). For any such A we shall also write $A^\sim := A^\circ/\mathfrak{m}A^\circ$.

2.2.5. Another possible source of confusion is the following situation. Let A be a normal domain of topologically finite type over K, and endow A with its supremum norm $|\cdot|_{\mathrm{sup}}$; let also $A \to B$ be an injective finite ring homomorphism. According to proposition 1.1.11, B is endowed with its spectral seminorm $|\cdot|_{\mathrm{sp}}$; on the other hand, B can also be endowed with its supremum seminorm and the problem arises whether these two seminorms coincide. According to [6, §3.8.1, Prop.7], this turns out to be the case, provided that B is torsion-free as an A-module.

Lemma 2.2.6. *Let A be a reduced affinoid K-algebra of topologically finite type. Let $U \subset X := \mathrm{Spa}\, A$ be an affinoid subdomain. Then $\mathscr{O}_X(U)$ is reduced.*

Proof. It suffices to show that, for every maximal ideal $\mathfrak{q} \subset B := \mathscr{O}_X(U)$, the localization $B_\mathfrak{q}$ is reduced. However, by a theorem of Kiehl, $B_\mathfrak{q}$ is excellent (see [11, Th.1.1.3] for a proof), hence it suffices to show that the $\mathfrak{q}$-adic completion $B^\wedge_\mathfrak{q}$ of $B_\mathfrak{q}$ is reduced. Let $\mathfrak{p} := \mathfrak{q} \cap A$; since U is a subdomain in X, the natural map $A \to B$ induces an isomorphism of complete local rings $A^\wedge_\mathfrak{p} \simeq B^\wedge_\mathfrak{q}$, so we are reduced to showing that $A^\wedge_\mathfrak{p}$ is reduced, which holds because $A_\mathfrak{q}$ is reduced and excellent (again by Kiehl's theorem). □

2.2.7. For every $a, b \in \Gamma_K$ with $a \leq b$, we denote by $\mathbb{D}(a)$ the disc of radius a, and by $\mathbb{D}(a, b)$ the annulus of radii a and b. Say that $a = |\alpha|$ and $b = |\beta|$ for $\alpha, \beta \in K^\times$; then

$$\mathbb{D}(a, b) := \mathrm{Spa}\, A(a, b) \quad \text{and} \quad \mathbb{D}(a) := \mathrm{Spa}\, A(a)$$

where $A(a, b)$ (resp. $A(a)$) is the affinoid K-algebra of topologically finite type such that

$$A(a, b)^\triangleright := K\langle \alpha/\xi, \xi/\beta \rangle \qquad \text{(resp. } A(a)^\triangleright := K\langle \xi/\alpha \rangle\text{)}.$$

Hence $A(a, b)^+ = A(a, b)^\circ = K^+\langle \alpha/\xi, \xi/\beta \rangle$ and $A(a)^+ = A(a)^\circ = K^+\langle \xi/\alpha \rangle$.

2.2.8. Let X be any adic space locally of finite type over $\mathrm{Spa}\, K$ and with $\dim X = 1$. The points of X fall into three distinct classes, according to whether: (I) they admit neither a proper generalization nor a proper specialization, or (II) they admit a proper specialization, or else (III) they have a proper generalization. For every point $x \in X$ of class (III), we shall denote by $x^\flat$ the unique generization of x in X, so $x^\flat$ is a point of class (II). The value group Γ_x of $|\cdot|_x$ admits a natural decomposition (see [31, §1.1 and Cor.5.4])

$$\Gamma_x \simeq \Gamma_x^{\mathrm{div}} \oplus \langle\gamma_0\rangle$$

where Γ_x^{div} is the maximal divisible subgroup contained in Γ_x, and $\langle\gamma_0\rangle \simeq \mathbb{Z}$ is the subgroup generated by the element γ_0 uniquely characterized as the largest element of the subset $\Gamma_x^+ \setminus \{1\}$ (notation of (1.1.1)).

2.2.9. For instance, take $X := (\mathbb{A}^1_K)^{\mathrm{ad}}$, the analytification of the affine line. The topological space underlying $(\mathbb{A}^1_K)^{\mathrm{ad}}$ consists of the equivalence classes of continuous valuations $v : K[\xi] \to \Gamma_v$ extending the valuation of K. These valuations are described in [31, §5]: to the class (I) belong *e.g.* the height one valuations of the form

$$f(\xi) \mapsto |f(x)| \qquad \text{for all } f(\xi) \in K[\xi]$$

where $x \in K = \mathbb{A}^1_K(K)$ is any element (these are the K-rational points of $(\mathbb{A}^1_K)^{\mathrm{ad}}$). The valuations of classes (II) and (III) are all of height respectively one and two. The elements of these classes admit a uniform description, that we wish to explain. To this aim we consider an imbedding of ordered fields

$$(\mathbb{R}, <) \hookrightarrow (\mathbb{R}(\varepsilon), <)$$

where $\mathbb{R}(\varepsilon)$ is a purely transcendental extension of $\mathbb{R}$, generated by an element ε such that $0 < \varepsilon < r$ for every real number $r > 0$. One can view $\mathbb{R}(\varepsilon)$ as a subfield of the ordered field of hyperreal numbers ${}^*\mathbb{R}$ (see [27]). For every $x \in K$, every real number $r > 0$ and every $\omega \in \{1, 1-\varepsilon, 1/(1-\varepsilon)\} \subset \mathbb{R}(\varepsilon)$ consider the valuation

$$|\cdot|_{r\cdot\omega} : K[\xi] \to \mathbb{R}(\varepsilon) \quad : \quad \sum_{i=0}^n a_i(\xi - x)^n \mapsto \max(|a_i| \cdot r^i \cdot \omega^i \mid i = 0, \dots, n).$$

If $\omega = 1$, this is the usual Gauss (sup) norm attached to the disc of radius r centered at the point x; this is a valuation of height one. For $\omega \neq 1$ we get a valuation which should be thought of as the sup norm on a disc of radius $r \cdot \omega$, again centered at x; this new kind of valuation is a specialization of $|\cdot|_r$, and indeed all the specializations of the latter occur in this manner. If $r \notin \Gamma_K$, then $|\cdot|_r$ belongs to the class (I); in this case the valuations $|\cdot|_{r\cdot\omega}$ are all equivalent, regardless of ω, and therefore they induce the same point of $(\mathbb{A}^1_K)^{\mathrm{ad}}$. If $r \in \Gamma_K$ then $|\cdot|_r$ is in the class (II); in this case the $|\cdot|_{r\cdot\omega}$ for $\omega \neq 1$ are two inequivalent valuations of height two, hence of class (III), and indeed all valuations of class (III) arise in this way.

2.2.10. Let $a, b \in \Gamma_K$ with $a \leq b$. For $r \in (a, b] \cap \Gamma_K$, the valuation $|\cdot|_{r(1-\varepsilon)}$ extends to $A(a, b)$ by continuity; if moreover $r > a$, then the point of $(\mathbb{A}^1_K)^{\mathrm{ad}}$ corresponding to this valuation lies in the open subdomain $\mathbb{D}(a, b)$. This point shall be denoted henceforth by $\eta(r)$, and to lighten notation we shall write $\kappa(r)$ (resp. $\kappa(r^\flat)$) for the residue field of $\eta(r)$ (resp. of $\eta(r)^\flat$). Notice that $\kappa(r)$ is also the same as the stalk $\mathscr{O}_{\mathbb{D}(a,b),\eta(r)}$.

Likewise, if $r \in [a, b) \cap \Gamma_K$, the valuation $|\cdot|_{r/(1-\varepsilon)}$ determines a point $\eta'(r)$ of $(\mathbb{A}^1_K)^{\mathrm{ad}}$ that lies in the open subdomain $\mathbb{D}(a, b)$; the residue field of $\eta'(r)$ shall be denoted $\kappa'(r)$. Notice that $\eta(r)^\flat = \eta'(r)^\flat$.

2.2.11. Let $f : X \to \mathbb{D}(a,b)$ be a finite and flat morphism of affinoid adic spaces of degree d, and suppose that X is reduced (*i.e.* $X = \mathrm{Spa}\, B$ where B is a reduced flat affinoid algebra of rank d as an $A(a,b)$-module). For every $r \in (a,b] \cap \Gamma_K$, we set :

$$\mathscr{B}(r) := (f_*\mathscr{O}_X)_{\eta(r)}$$

which is a reduced finite $\kappa(r)$-algebra, in view of lemma 2.2.6. We endow $\mathscr{B}(r)$ with the spectral norm $|\cdot|_{\mathrm{sp},\eta(r)}$ relative to the valuation $|\cdot|_{\eta(r)}$; it follows that

$$\mathscr{B}(r)^+ = (f_*\mathscr{O}_X^+)_{\eta(r)}.$$

Lemma 2.2.12. *In the situation of* (2.2.11)*, let $y \in \mathbb{D}(a,b)$ any point. Then $f^{-1}(y)$ is the set of all the valuations on $(f_*\mathscr{O}_X)_y$ that extend the valuation $|\cdot|_y$ corresponding to y.*

Proof. Let $X = \mathrm{Spa}(B^\triangleright, B^\circ)$; the topology of $B^\triangleright$ is the $A(a,b)$-module topology, *i.e.* the unique one such that the family of $A(a,b)^\circ$-submodules $(\pi B^\circ \mid \pi \in \mathfrak{m} \setminus \{0\})$ is a fundamental system of neighborhoods of 0. Let $B_y := B^\triangleright \otimes_{A(a,b)^\triangleright} \mathscr{O}_y = (f_*\mathscr{O}_X)_y$; similarly B_y has a well-defined $\mathscr{O}_y$-module topology and the fibre $f^{-1}(y)$ consists of the continuous valuations $|\cdot|'$ on B_y extending the valuation $|\cdot|_y$, and such that

$$|s|' \leq 1 \qquad \text{for all } s \in B^\circ. \tag{2.2.13}$$

Let $|\cdot|'$ be any valuation on B_y extending $|\cdot|_y$, and let $\mathfrak{p} \subset B_y$ be the support of $|\cdot|'$; the quotient topology on $E := B_y/\mathfrak{p}$ is the $\kappa(y)$-module topology, where $\kappa(y)$ is the residue field of y. However, let Γ_y and Γ_E be the value groups of $|\cdot|_y$ and respectively $|\cdot|_E$; since $[\Gamma_E : \Gamma_y]$ is finite, it is easy to see that $|\cdot|'$ is continuous. Hence, continuity holds for all $|\cdot|'$ extending $|\cdot|_y$, and since anyway B° is the integral closure of $A(a,b)^\circ$ in $B^\triangleright$, the same goes for condition (2.2.13). □

2.2.14. In the situation of (2.2.11), let $x \in X$ be a point of class (III). The valuation $|\cdot|_x$ is an extension of the valuation $|\cdot|_{f(x)} : \kappa(f(x)) \to \mathbb{R}(\varepsilon)$, hence its value group can be realized inside the multiplicative group of the field $\mathbb{R}((1-\varepsilon)^{1/d!})$, which is an algebraic extension of $\mathbb{R}(\varepsilon)$ of degree $d!$, admitting a unique ordering extending the ordering of $\mathbb{R}(\varepsilon)$ (again, one can think of all this as taking place inside the hyperreal numbers; of course, there is no real need to introduce this auxiliary field: it is nothing more than a suggestive notational device). In terms of the decomposition of (2.2.8), we have $\Gamma_x^{\mathrm{div}} \subset \mathbb{R}^\times_{>0}$ and $\langle\gamma_0\rangle \subset \{(1-\varepsilon)^i \mid i \in \frac{1}{d!}\mathbb{Z}\}$. We shall consider the two projections:

$$\Gamma_x \to \Gamma_x^{\mathrm{div}} \quad : \quad \gamma \mapsto \gamma^\flat \qquad \text{and} \qquad \Gamma_x \to \frac{1}{d!}\mathbb{Z} \quad : \quad \gamma \mapsto \gamma^\natural$$

where $\gamma^\natural$ is characterized by the identity

$$(1-\varepsilon)^{\gamma^\natural} \cdot \gamma^\flat = \gamma \qquad \text{for every } \gamma \in \Gamma_x.$$

Sometimes it is more natural to use an additive (rather than multiplicative) notation; in order to switch from one to the other, of course one takes logarithms. Hence we define :

$$\log\gamma := \log\gamma^\flat - \gamma^\natural \cdot \varepsilon \in \mathbb{R} + \varepsilon\mathbb{R} \qquad \text{for every } \gamma \in \Gamma_x. \tag{2.2.15}$$

The composition

$$B \to \Gamma_x^{\mathrm{div}} \cup \{0\} \quad : \quad s \mapsto |s|_x^\flat$$

is a continuous rank one valuation of B and determines the unique generization $x^\flat$ of x in X. If we view $\mathbb{R}((1-\varepsilon)^{1/d!})$ as a subfield of the hyperreal numbers, then the projection $|s|_x^\flat$ corresponds to the shadow of the bounded hyperreal $|s|_x$.

2.2.16. The ring $\mathscr{B}(r)$ is a product of finite field extensions $F_1 \times \cdots \times F_k$ of $\mathscr{O}_{\eta(r)}$, and the factors F_j are in natural bijective correspondence with the elements of the fibre $f^{-1}(\eta(r)^\flat)$ (see also [30, Prop.1.5.4]). Moreover, $\mathscr{B}(r)^+$ decomposes as the product $F_1^+ \times \cdots \times F_k^+$, where F_j^+ is the integral closure of $\kappa(r)^+$ in F_j. The valuation ring $\kappa(r)^+$ is not henselian, hence it may occur that F_j^+ is not a valuation ring, but only a normed $\kappa(r)^+$-algebra; that happens precisely when there are distinct points $x, y \in f^{-1}(\eta(r))$ such that $x^\flat = y^\flat$ (see example 2.3.13).

Lemma 2.2.17. *In the situation of* (2.2.11), *suppose furthermore that the morphism f is generically étale. Then, for every $r \in (a, b] \cap \Gamma_K$ we have:*

(i) *The normed ring $(\mathscr{B}(r)^+, |\cdot|_{\mathrm{sp},\eta(r)})$ is a free cartesian $\kappa(r)^+$-module of rank d.*
(ii) $|s|_{\mathrm{sp},\eta(r)} = \max(|s|_x \mid x \in f^{-1}(\eta(r)))$ *for all $s \in \mathscr{B}(r)$.*
(iii) *Let us view $\mathscr{B}(r)^+$ as a submodule of the normed cartesian module $(\mathscr{B}(r), |\cdot|_{\mathrm{sp},\eta(r)})$, so that the value $|\mathscr{B}(r)^+|_{\mathrm{sp},\eta(r)}$ is defined (notation of* (1.2.25)*). Then:*

$$2|\mathscr{B}(r)^+|^\natural_{\mathrm{sp},\eta(r)} = \deg(f) - \sharp f^{-1}(\eta(r))$$

where $\sharp f^{-1}(\eta(r))$ denotes the cardinality of the fibre $f^{-1}(\eta(r))$.
(iv) *If $R \subset \mathscr{B}(r)^+$ is a finite $\kappa(r)^+$-algebra such that $R \otimes_{K^+} K = \mathscr{B}(r)$, then $R = \mathscr{B}(r)^+$ if and only if $R/\mathfrak{m}R$ is a reduced $K^\sim$-algebra.*

Proof. (i): under the current assumptions, the field extension $\kappa(r)^\wedge \subset \kappa(x)^\wedge$ is finite (see remark 2.2.3(i)) and separable. Let us set :

$$\mathscr{B}(r)^{\wedge+} := \mathscr{B}(r)^+ \otimes_{\kappa(r)^+} \kappa(r)^{\wedge+} \qquad \mathscr{B}(r)^\wedge := \mathscr{B}(r) \otimes_{\kappa(r)} \kappa(r)^\wedge.$$

Since $\mathscr{B}(r)$ is an étale $\kappa(r)$-algebra, $\mathscr{B}(r)^\wedge$ is an étale $\kappa(r)^\wedge$-algebra, especially it is reduced; by flatness, $\mathscr{B}(r)^{\wedge+}$ is a subalgebra of $\mathscr{B}(r)^\wedge$, hence it is reduced as well. Hence lemma 1.2.3(ii) applies, and reduces to showing that $(\mathscr{B}(r)^{\wedge+}, |\cdot|^\wedge_{\mathrm{sp}})$ is a free cartesian $\kappa(r)^{\wedge+}$-module of rank d. Furthermore, proposition 1.3.2(iii) implies that $\mathscr{B}(r)^{\wedge+}$ is normal, hence it is the direct product

$$\mathscr{B}(r)^{\wedge+} = L_1^+ \times \cdots \times L_k^+ \tag{2.2.18}$$

of finitely many normal domains, and each L_i^+ is the integral closure of $\kappa(r)^{\wedge+}$ in a finite algebraic extension L_i of $\kappa(r)^\wedge$. Notice as well, that $\kappa(r^\flat)^{\wedge+}$ is henselian, since it is complete and of rank one; hence $L_i^+ \otimes_{\kappa(r)^{\wedge+}} \kappa(r^\flat)^{\wedge+}$ is a valuation ring whose valuation extends $|\cdot|^\flat_{\eta(r)}$. On the other hand, by lemma 2.2.12 the points of $x \in f^{-1}(\eta(r))$ correspond to the valuations $|\cdot|_x$ on $\mathscr{B}(r)$ that extend $|\cdot|_{\eta(r)}$; these are also the valuations on $\mathscr{B}(r)^\wedge$ that extend $|\cdot|^\wedge_{\eta(r)}$. Hence, the decomposition (2.2.18) induces a partition

$$f^{-1}(\eta(r)) = \Sigma_1 \cup \cdots \cup \Sigma_k$$

where, for each $i \leq k$, Σ_i is the set of valuations of L_i that extend $|\cdot|^\wedge_{\eta(r)}$.

By lemma 1.1.17(ii), we are reduced to showing :

Claim 2.2.19. For every $i \leq k$, let $|\cdot|_{\mathrm{sp},i}$ be the spectral norm of the $\kappa(r)^\wedge$-algebra L_i; then $(L_i^+, |\cdot|_{\mathrm{sp},i})$ is a $\kappa(r)^{\wedge+}$-cartesian module.

Proof of the claim. It suffices to show that the finite field extension $\kappa(r)^\wedge \subset L_i$ fulfills conditions (a)–(d) of proposition 1.2.8. However, condition (a) is none else than [31, Lemma 5.3(ii)]. Next, say that $\Sigma_i = \{x_1, \dots, x_l\}$, and let $\Gamma_1, \dots, \Gamma_l$ be the value groups of the residue fields $\kappa(x_i)$. By inspecting the construction, one sees easily that $x_i^\flat = x_j^\flat$ for every $i, j \leq l$, which means that the subgroup $\Delta_{ij} \subset \Gamma_i$ defined as in (1.2.7), is the unique convex subgroup corresponding to $x_i^\flat$, so (d) holds, and also (b) is clear. Finally, (c) follows easily from [25, Cor.5.4 and Prop.1.2(iii)]. $\diamondsuit$

(ii): In view of lemma 1.1.17(i.a), it suffices to show the analogous identity for

$$(\mathscr{B}(r)^{\wedge h}, |\cdot|^{\wedge h}_{\mathrm{sp},\eta(r)}) := (\mathscr{B}(r), |\cdot|_{\mathrm{sp},\eta(r)}) \otimes_{\kappa(r)} \kappa(r)^{\wedge h}.$$

By the proof of (i) we know that $\mathscr{B}(r)^{\wedge h} = \prod_{x \in f^{-1}(\eta(r))} \kappa(x)^{\wedge h}$, so the assertion follows from lemma 1.1.17(ii),(iii).

(iii): In light of lemma 1.2.3(i) it suffices to show the same identity for $|\mathscr{B}(r)^{\wedge h+}|^{\wedge h\natural}_{\mathrm{sp}}$; however, from [31, Prop.1.2(iii) and Cor.5.4] we deduce that

$$|\kappa(x)^{\wedge h+}|^{\wedge h\natural}_{x} = \frac{[\kappa(x)^{\wedge h} : \kappa(r)^{\wedge h}]}{2}$$

for every $x \in f^{-1}(\eta(r))$; then the assertion follows easily.

(iv): Suppose first that $R/\mathfrak{m}R$ is reduced. According to (i), R is a finitely generated submodule of a free $\kappa(r)^+$-module of finite rank; hence it is free as a $\kappa(r)^+$-module. Hence every $x \in R$ can be written in the form $x = ay$ for some $a \in K^+$ and an element $y \in R$ whose image in $R/\mathfrak{m}R$ does not vanish. It follows that every $x \in \mathscr{B}(r)$ can be written in the form $x = ay$ for some $a \in K$ and some $y \in R \setminus \mathfrak{m}R$. Let now $x \in \mathscr{B}(r)^+$, and suppose that $x = ay$ for some $y \in R \setminus \mathfrak{m}R$ and $a \in K$ with $|a| > 1$; clearly x is integral over R, so we can write

$$x^n + b_1 x^{n-1} + \cdots + b_n = 0$$

for some $b_1, \dots, b_n \in R$. Hence

$$y^n + b_1 a^{-1} y^{n-1} + \cdots + b_n a^{-n} = 0.$$

In other words, $y^n \in \mathfrak{m}R$, whence $y \in \mathfrak{m}R$, since $R/\mathfrak{m}R$ is reduced; the contradiction shows that $R = \mathscr{B}(r)^+$. Conversely, suppose that $R = \mathscr{B}(r)^+$ and let $x \in R$ whose image in $R/\mathfrak{m}R$ is nilpotent; then $x^n \in \mathfrak{m}R$ for $n \in \mathbb{N}$ large enough, say $x^n = ay$ for some $a \in \mathfrak{m}$ and $y \in R$. We can write $a = b^n c$ for $b, c \in \mathfrak{m}$, so $(x/b)^n = cy \in R$, so $x/b \in R$, since the latter is integrally closed in $R \otimes_{K^+} K$. Finally, $x \in \mathfrak{m}R$, as claimed. □

Lemma 2.2.20. *In the situation of* (2.2.11), *let* $r \in (a, b] \cap \Gamma_K$, $U \subset \mathbb{D}(a, b)$ *an open neighborhood of* $\eta(r)$, $s \in \Gamma(U, f_*\mathscr{O}_X)$ *and set* $k := |s|^{\natural}_{\mathrm{sp},\eta(r)} \in \frac{1}{d!}\mathbb{Z}$. *Then there exists* $r' \in (a, r)$ *such that:*

(i) $\eta(t) \in U$ *for every* $t \in (r', r] \cap \Gamma_K$.
(ii) $|s|_{\mathrm{sp},\eta(t)} = |s|_{\mathrm{sp},\eta(r)} \cdot (t/r)^k$ *whenever* $t \in (r', r] \cap \Gamma_K$.

Proof. (i) is obvious. In order to prove (ii), consider an integral equation

$$s^n + g_1 s^{n-1} + \cdots + g_n = 0$$

where $g_i \in \Gamma(U, \mathscr{O}_{\mathbb{D}(a,b)})$ for every $i \leq n$. It is easy to see that the assertion for s will follow once the same assertion is known for the sections $g_1, \dots, g_n$. Hence, we are reduced to the case $X = \mathbb{D}(a, b)$. We can also assume that there exist $\alpha_1, \dots, \alpha_m \in K$ such that $|\alpha_i| = r$ for every $i \leq m$ and

$$U = \mathbb{D}(r', r) \setminus \bigcup_{i=1}^{m} \mathbb{E}_i$$

where $\mathbb{E}_i = \{p \in \mathbb{D}(r', r) \mid |\xi - \alpha_i|_p < r\}$ (so each $\mathbb{E}_i$ is a closed subset of $\mathbb{D}(r', r)$). Then, in view of [24, Prop.2.8] we have a Mittag-Leffler decomposition

$$s = s_0 + s_1 + \sum_{i=1}^{m} h_i$$

where

$$s_0 = \sum_{n=0}^{\infty} a_n \xi^n \qquad s_1 = \sum_{n=1}^{\infty} b_n \xi^{-n} \qquad h_i = \sum_{n=1}^{\infty} c_{i,n} (\xi - \alpha_i)^{-n} \quad (i = 1, \dots, m)$$

for coefficients $a_n, b_n, c_{i,n} \in K$ subject to the conditions:

$$\lim_{n\to\infty} r^n|a_n| = \lim_{n\to\infty} r'^{-n}|b_n| = \lim_{n\to\infty} r^{-n}|c_{i,n}| = 0.$$

Recalling the standard power series identity:

$$(\xi - \alpha_i)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-\alpha_i)^{-n-k}\xi^k$$

we deduce that

$$|s_1|_{\eta(t)} = \sup_{n>0} |b_n| \cdot t^{-n}(1-\varepsilon)^{-n} \qquad \text{for every } t \in (r'r] \cap \Gamma_K$$

and

$$\left| s_0 + \sum_{i=1}^m h_i \right|_{\eta(t)} = \sup_{k\in\mathbb{N}} |d_k| \cdot t^k(1-\varepsilon)^k \quad \text{for every } t \in (r'r] \cap \Gamma_K.$$

where

$$d_k = a_k + \sum_{i=1}^m \sum_{n=1}^{\infty} \binom{-n}{k} (-\alpha_i)^{-n-k} c_{i,n} \quad \text{for every } k \in \mathbb{N}.$$

Notice that $|s_1|_{\eta(t)} \neq |s_0 + \sum_{i=1}^m h_i|_{\eta(t)}$ for every $t \in (r', r] \cap \Gamma_K$, therefore

$$|s|_{\eta(t)} = \max\left(|s_1|_{\eta(t)}, \left| s_0 + \sum_{i=1}^m h_i \right|_{\eta(t)} \right)$$

and hence it suffices to prove assertion (ii) separately for s_1 and $s_0 + \sum_{i=1}^m h_i$.

The coefficients d_k enjoy the following property. There exists a smallest $k_0 \in \mathbb{N}$ such that

$$|d_{k_0}| = \sup_{k\in\mathbb{N}} |d_k| \cdot r^k.$$

Namely, k_0 is the unique integer such that

$$\left| s_0 + \sum_{i=1}^m h_i \right|_{\eta(r)} = |d_{k_0}| \cdot r^{k_0}(1-\varepsilon)^{k_0}.$$

Thus, when evaluating $|s_0 + \sum_{i=1}^m h_i|_{\eta(t)}$ for $t < r$, we can disregard all the terms $d_k\xi^k$ for $k > k_0$, since their norms decrease faster than that of the leading term $d_{k_0}\xi^{k_0}$. There remain to consider the finitely many monomials $d_0, d_1\xi, \dots, d_{k_0-1}\xi^{k_0-1}$; however, it is clear from the definition of k_0 that the norms of these terms are still strictly smaller than the norm of the leading term $d_{k_0}\xi^{k_0}$, as long as t is sufficiently close to r. Thus, we can replace the section $s_0 + \sum_{i=1}^m h_i$ by its leading monomial, for which the sought assertion trivially holds; an analogous, though easier argument also works for s_1 : we leave the details to the reader. □

2.3. **Convexity and piecewise linearity of the discriminant function.** The assumptions and notations are as in (2.2). The statements proven so far make use of only a few relatively simple local properties of the sheaf $f_*\mathscr{O}^+_X$; nevertheless, they would already suffice to prove most of the forthcoming proposition 2.3.17. However, in order to show theorem 2.3.35, it will be necessary to cast a closer look at the ring of global sections of $f_*\mathscr{O}^+_X$; the following lemmata 2.3.1, 2.3.2 and proposition 2.3.5 will provide us with everything we need.

Lemma 2.3.1. *Let $A \to B$ be a finite morphism of K-algebras of topologically finite type. Then A° is of topologically finite presentation over K^+ and B° is a finitely presented A°-module.*

Proof. By [6, §6.4.1, Cor.5] we know that B° is a finite A°-module; moreover, applying *loc.cit.* to an epimorphism $K\langle T_1, \dots, T_n\rangle \to A$ we deduce that A° is finite over $K^+\langle T_1, \dots, T_n\rangle$. By [25, Prop.7.1.1(i)] we deduce that A° is finitely presented as a $K^+\langle T_1, \dots, T_n\rangle$-module, and also that B° is finitely presented over A°. □

Lemma 2.3.2. *Let B be a flat, reduced K^+-algebra of topologically finite type, and set $A := B \otimes_{K^+} K$. Then $B = A^\circ$ if and only if $B/\mathfrak{m}B$ is reduced.*

Proof. (Notice that the assertion is a global version of lemma 2.2.17(iv), and indeed its proof is analogous to that of the former.)

Claim 2.3.3. B/aB is a free K^+/aK^+-module for every $a \in \mathfrak{m}$.

Proof of the claim. By [25, Prop.7.1.1(i)] we have $B \simeq K^+\langle T_1, \dots, T_r\rangle/I$ for some $r \geq 0$ and a finitely generated ideal I. It follows that $B/aB \simeq K^+/aK^+[T_1, \dots, T_r]/J$, where J is the image of I. We can write $K^+/aK^+ = \bigcup_{\lambda\in\Lambda} R_\lambda$, the filtered union of its noetherian local subalgebras R_λ. By [21, Ch.IV, Prop.8.5.5] we can find $\lambda \in \Lambda$ and a finitely presented flat R_λ-algebra B_λ such that $B/aB \simeq B_\lambda \otimes_{R_\lambda} K^+/aK^+$. It suffices to show that B_λ is a free R_λ-module; however, since R_λ is artinian, this follows from [41, Th.7.9]. ◊

Claim 2.3.4. A° is the integral closure of B in A.

Proof of the claim. Choose a continuous surjection $\phi : C := K^+\langle T_1, \dots, T_r\rangle \to B$. It suffices then to notice that $C = (C \otimes_{K^+} K)^\circ$ and apply [6, §6.3.4, Prop.1]. ◊

By claim 2.3.3, every $x \in B \setminus \{0\}$ can be written in the form $x = ay$ for some $a \in K^+$ and an element $y \in B$ whose image in $B/\mathfrak{m}B$ does not vanish. It follows that every $x \in A\setminus\{0\}$ can be written in the form $x = ay$ for some $a \in K$ and some $y \in B \setminus \mathfrak{m}B$. After these remarks, one can proceed as in the proof of lemma 2.2.17(iv): the details shall be entrusted to the reader. □

Proposition 2.3.5. *Let $(F, |\cdot|_F)$ be a complete algebraically closed valued field extension of K, such that $|\cdot|_F$ is a valuation of rank one. Let A be a normal domain of topologically finite type over K, such that $A^\sim$ is a principal ideal domain, B a finite, reduced and flat A-algebra, and $g \in A$ such that $|g|_{\mathrm{sup}} = 1$. Set $A_F := A\widehat{\otimes}_K F$. Then :*

(i) *B° is a free A°-module of finite rank.*
(ii) $B\langle g^{-1}\rangle^\circ = B^\circ \otimes_{A^\circ} A\langle g^{-1}\rangle^\circ$.
(iii) $(B \otimes_A A_F)^\circ = B^\circ \otimes_{A^\circ} A_F^\circ$.

Proof. To start out, let us endow A and B with their supremum norms; then by (2.2.4) and [6, §3.8.1, Prop.7], we have $A^\circ = A^+$ and $B^\circ = B^+$. By lemma 2.3.1 we deduce that A^+ is of topologically finite type over K^+ and B^+ is finitely presented over A^+.

Claim 2.3.6. $B^\sim$ is free of finite rank over $A^\sim$.

Proof of the claim. By the foregoing we know already that $B^\sim$ is finite over $A^\sim$, hence it suffices to show that $B^\sim$ is torsion-free as an $A^\sim$-module. However, under the current assumptions the norm $|\cdot|_{\mathrm{sup}}$ on A is a valuation ([6, §6.2.3, Prop.5]). It follows that

$$|b|_{\mathrm{sp}} = \max_v v(b) \qquad \text{for all } b \in B \tag{2.3.7}$$

where v ranges over the finitely many extensions of the supremum valuation of A to B ([6, §3.3.1, Prop.1]). For each such v, let $\mathrm{supp}(v) := v^{-1}(0)$, which is a prime ideal of B, and denote by $B_v \subset \mathrm{Frac}(B/\mathrm{supp}(v))$ the valuation ring of the valuation induced by v on $B/\mathrm{supp}(v)$. Since Γ_K is divisible, it is easy to see that $\mathfrak{m}B_v$ is the maximal ideal of B_v. From (2.3.7) it is clear that $B^\sim \subset \prod_v B_v/\mathfrak{m}B_v$. Finally, for every v the field $B_v/\mathfrak{m}B_v$ is a finite extension of $\mathrm{Frac}(A^\sim)$, especially it is torsion-free over $A^\sim$, and the same holds then for $B^\sim$. ◊

From claim 2.3.6 and [21, Ch.IV, Prop.8.5.5] it follows that there exists $\pi \in \mathfrak{m}$ such that $B^+/\pi B^+$ is flat over $A^+/\pi A^+$. In view of [25, Lemma 5.2.1] we conclude that B^+ is flat, hence projective over A^+. Finally, a standard application of Nakayama's lemma shows that any lifting of a basis of $B^\sim$ is a basis of the A^+-module B^+, which proves (i).

Claim 2.3.8. The ring $C := B^\circ \otimes_{A^\circ} A\langle g^{-1}\rangle^\circ$ is reduced.

Proof of the claim. From (i) we deduce that the natural map

$$B^\circ \otimes_{A^\circ} A\langle g^{-1}\rangle^\circ \to B\langle g^{-1}\rangle = B^\circ \otimes_{A^\circ} A\langle g^{-1}\rangle$$

is injective. Hence it suffices to show that $B\langle g^{-1}\rangle$ is reduced, which holds by lemma 2.2.6. $\quad\diamondsuit$

In view of claim 2.3.8 and lemma 2.3.2, assertion (ii) will follow once we know that $C/\mathfrak{m}_k C$ is reduced. However, the latter is isomorphic to $B^\sim \otimes_{A^\sim} A^\sim[\overline{g}^{-1}]$, where $\overline{g} \in A^\sim$ is the image of g ([6, §7.2.6, Prop.3]). Again lemma 2.3.2 ensures that $B^\sim$ is reduced.

(iii): According to [11, Lemma 3.3.1.(1)], $B_F := B \otimes_A A_F = B^\circ \otimes_{A^\circ} A_F$ is reduced. From (i) we deduce that $D := B^\circ \otimes_{A^\circ} A_F^\circ$ is a subalgebra of B_F, so it is reduced as well. Hence in order to prove (iii) it suffices, by lemma 2.3.2, to show that $D/\mathfrak{m}_F D$ is reduced (where $\mathfrak{m}_F$ is the maximal ideal of F^+). However, $D/\mathfrak{m}_F D \simeq B^\sim \otimes_{K^\sim} F^\sim$ and the extension $K^\sim \to F^\sim$ is separable, so everything is clear. $\square$

2.3.9. Let $a, b \in \Gamma_K$ with $a < b$ and $f : X \to \mathbb{D}(a,b)$ a finite, flat and generally étale morphism, say of degree d. For every $r \in [a,b] \cap \Gamma_K$ we define:

$$\mathscr{B}(r)^\flat := (f_*\mathscr{O}_X^+)_{\eta(r)^\flat}.$$

It follows easily from proposition 2.3.5(ii) that

$$\mathscr{B}(r)^\flat = \mathscr{B}(r)^+ \otimes_{\kappa(r)^+} \kappa(r^\flat)^+ \qquad \text{for every } r \in (a,b] \cap \Gamma_K \tag{2.3.10}$$

(notation of (2.2.10) and (2.2.11)).

2.3.11. The apparent asymmetry between the values a and b can be easily resolved. Indeed, let us consider the isomorphism

$$q : \mathbb{D}(1/b, 1/a) \to \mathbb{D}(a,b) \quad : \quad \xi \mapsto \xi^{-1}$$

and let $g := q^{-1} \circ f$. For every $r \in (1/b, 1/a]$, the image $q(\eta(r))$ is the point $\eta'(1/r)$ (notation of (2.2.10)). Hence, $g^* : \mathscr{O}_{\mathbb{D}(1/b,1/a)} \to g_*\mathscr{O}_X$ endows $(f_*\mathscr{O}_X^+)_{\eta'(a)}$ with a structure of $\kappa(1/a)^+$-algebra, and since $\eta'(a)^\flat = \eta(a)^\flat$, we deduce that an identity analogous to (2.3.10) holds also for $r = a$, provided we replace $\eta(a)$ by $\eta'(a)$. Especially, since – according to lemma 2.2.17(i) – the stalk $\mathscr{B}(r)^+$ is a free $\kappa(r)^+$-module of rank d, we see that $\mathscr{B}(r)^\flat$ is a free $\kappa(r^\flat)^+$-module of rank d for every $r \in [a,b] \cap \Gamma_K$.

2.3.12. Now, as f is generically étale, the trace forms $\mathrm{Tr}_{\mathscr{B}(r)^\flat/\kappa(r^\flat)^+}$ and $\mathrm{Tr}_{\mathscr{B}(r)^+/\kappa(r)^+}$ induce the same perfect pairing after tensoring with $\kappa(r^\flat)$. We set

$$\mathfrak{d}_f^+(r) := \mathfrak{d}_{\mathscr{B}(r)^+/\kappa(r)^+} \in \kappa(r)^+ \qquad \text{for every } r \in (a,b] \cap \Gamma_K.$$

(respectively:

$$\mathfrak{d}_f^\flat(r) := \mathfrak{d}_{\mathscr{B}(r)^\flat/\kappa(r^\flat)^+} \in \kappa(r^\flat)^+ \qquad \text{for every } r \in [a,b] \cap \Gamma_K.$$

Notation of (2.1).) Since $\mathfrak{d}_f^\flat(r)$ is well defined up to the square of an invertible element of $\kappa(r^\flat)^+$, the real-valued function:

$$\delta_f : [\log 1/b, \log 1/a] \cap \log \Gamma_K \to \mathbb{R}_{\geq 0} \qquad -\log r \mapsto -\log |\mathfrak{d}_f^\flat(r)|_{\eta(r)^\flat}$$

is well defined independently of all choices. Unless we have to deal with more than one morphism, we shall usually drop the subscript, and write δ, $\mathfrak{d}^\flat$ instead of δ_f, $\mathfrak{d}_f^\flat$. We call δ the *discriminant function* of the morphism f.

Example 2.3.13. Let $f : X \to \mathbb{D}(a, a^{-1})$ be a finite, flat and generically étale morphism, where $a := |\pi|$ for some $\pi \in \mathfrak{m}$. Using the Mittag-Leffler decomposition [24, Prop.2.8] one verifies easily that $A(a, a^{-1})^\circ = K^+\langle\pi/\xi, \xi/\pi\rangle \simeq K^+\langle S, T\rangle/(ST - \pi^2)$ (alternatively, one sees this via lemma 2.3.2). We set $h := \mathrm{Spa}(\psi_K) \circ f : X \to \mathbb{D}(1)$, where $\psi_K := \psi \otimes_{K^+} K$, with ψ defined as in example 2.1.12. A direct computation shows that

$$h^{-1}(\mathbb{D}(r, 1)) = f^{-1}(\mathbb{D}(a, a/r)) \cup f^{-1}(\mathbb{D}(r/a, a^{-1})) \qquad \text{for every } r \in (a, 1] \cap \Gamma_K.$$

Consequently:

$$\mathfrak{d}^\flat_h(r) = \mathfrak{d}^\flat_f(r/a) \cdot \mathfrak{d}^\flat_f(a/r) \qquad \text{whenever } r \in (a, 1] \cap \Gamma_K$$

and therefore

$$\delta_h(-\rho) = \delta_f(\rho - \log a) + \delta_f(\log a - \rho) \qquad \text{for } \rho \in (\log a, 0] \cap \log \Gamma_K. \tag{2.3.14}$$

Incidentally, let $\eta'(a) \in \mathbb{D}(1)$ be defined as in (2.2.10); it is easy to check that the preimage of $\eta'(a)$ in $\mathbb{D}(a, a^{-1})$ under $\mathrm{Spa}\,\psi_K$ is the subset $\{\eta(1), \eta'(1)\}$.

2.3.15. Let $f : [r, s] \to \mathbb{R}$ be a piecewise linear function; for every $\rho \in [r, s)$ we denote by $df/dt(\rho^+)$ the *right slope* of f at the point r, *i.e.* the unique real number α such that $f(\rho+x) = f(\rho) + \alpha x$ for every sufficiently small $x \geq 0$. Similarly we can define the *left slope* $df/dt(\rho^-)$ for every $\rho \in (r, s]$. More generally, the definition makes sense whenever f is defined on a dense subset of $[r, s]$.

Example 2.3.16. Let f and g be as in (2.3.11). Then

$$\delta_f(\rho) = \delta_g(-\rho) \qquad \text{and} \qquad \frac{d\delta_f}{dt}(\rho^-) = -\frac{d\delta_g}{dt}(-\rho^+)$$

for every $\rho \in (\log 1/b, \log 1/a] \cap \log \Gamma_K$.

Proposition 2.3.17. *With the notation of* (2.3.12)*, the function δ is piecewise linear; moreover:*

$$\frac{d\delta}{dt}(-\log r^+) = |\mathfrak{d}^+(r)|^\natural_{\eta(r)} - 2|\mathscr{B}(r)^+|^\natural_{\mathrm{sp},\eta(r)} \qquad \textit{for every } r \in (a, b] \cap \Gamma_K.$$

(Notation of (1.2.25)*.)*

Proof. By lemma 2.2.17(i) we can find an orthogonal basis $c_1, \dots, c_d$ of $(\mathscr{B}(r)^+, |\cdot|_{\mathrm{sp},\eta(r)})$ over $(\kappa(r)^+, |\cdot|_{\eta(r)})$. Obviously we have $|c_i|^\flat_{\mathrm{sp},\eta(r)} = 1$ for every $i = 1, \dots, d$; we set

$$\gamma_i := |c_i|^\natural_{\mathrm{sp},\eta(r)} \in \frac{1}{d!}\mathbb{Z} \qquad \text{for } i = 1, \dots, d. \tag{2.3.18}$$

For every $i, j \leq d$ we can find uniquely determined $m_{ij1}, \dots, m_{ijd} \in \kappa(r)^+$ such that

$$c_i \cdot c_j = \sum_{k=1}^{d} m_{ijk} c_k.$$

For every $i, j, k \leq d$ we set

$$x_{ijk} := |m_{ijk}|^\flat_{\eta(r)} \qquad \text{and} \qquad \mu_{ijk} := |m_{ijk}|^\natural_{\eta(r)} \tag{2.3.19}$$

so that $x_{ijk} \in [0, 1]$ and $\mu_{ijk} \in \mathbb{N}$. Since the c_i are orthogonal, it follows easily that

$$\mu_{ijk} \geq \gamma_i + \gamma_j - \gamma_k \qquad \text{whenever } x_{ijk} = 1. \tag{2.3.20}$$

Lemma 2.2.20(ii) implies that

$$\frac{m_{ijk}}{a^{\mu_{ijk}}}, \frac{c_i}{a^{\gamma_i}} \in \mathscr{B}(sr)^+ \qquad \text{for every } i, j, k = 1, \dots, d \tag{2.3.21}$$

whenever $a \in K^+$ is an element with $s := |a|$ sufficiently close to 1. Clearly we have

$$\frac{c_i}{a^{\gamma_i}} \cdot \frac{c_j}{a^{\gamma_j}} = \sum_k \frac{m_{ijk}}{a^{\gamma_i+\gamma_j-\gamma_k}} \cdot \frac{c_k}{a^{\gamma_k}} \tag{2.3.22}$$

which, in view of (2.3.20) and (2.3.21), means that

$$\mathscr{C}(sr) := \sum_{i=1}^{d} \kappa(sr)^+ \cdot \frac{c_i}{a^{\gamma_i}} \tag{2.3.23}$$

is a finite (unitary) subalgebra of $\mathscr{B}(sr)^+$ for every $s := |a|$ sufficiently close to 1.

Let $\mathscr{B}(r)^\sim := \mathscr{B}(r)^+ \otimes_{K^+} K^\sim$. The map $b \mapsto |b|^\natural_{\mathrm{sp},\eta(r)}$ induces a power-multiplicative norm on $\mathscr{B}(r)^\sim$, whence a filtration $\mathrm{Fil}^\bullet\mathscr{B}(r)^\sim$ defined by setting:

$$\mathrm{Fil}^h\mathscr{B}(r)^\sim := \{b \in \mathscr{B}(r)^\sim \mid |b|^\natural_{\mathrm{sp},\eta(r)} \geq h\} \qquad \text{for every } h \in \frac{1}{d!}\mathbb{Z}.$$

The filtration $\mathrm{Fil}^\bullet\mathscr{B}(r)^\sim$ restricts to a filtration $\mathrm{Fil}^\bullet\mathscr{O}^\sim_{\eta(r)}$ on $\mathscr{O}^\sim_{\eta(r)} := \kappa(r)^+ \otimes_{K^+} K^\sim$. The associated graded ring $\mathrm{gr}^\bullet\mathscr{B}(r)^\sim$ can be computed explicitly. Indeed, (2.3.18) shows that $\mathrm{gr}^\bullet\mathscr{B}(r)^\sim = \mathrm{gr}^\bullet\mathscr{O}^\sim_{\eta(r)}[\overline{c}_1,\dots,\overline{c}_d]$, where $\overline{c}_i$ is the class of c_i in $\mathrm{gr}^{\gamma_i}\mathscr{B}(r)^\sim$ for $i = 1,\dots,d$. Furthermore, for every $i,j \leq d$ we have the rule:

$$\overline{c}_i \cdot \overline{c}_j = \sum_{k=1}^{d} \beta_{ijk}\overline{c}_k$$

where β_{ijk} is determined as follows. If either $x_{ijk} < 1$ or $\mu_{ijk} > \gamma_i + \gamma_j - \gamma_k$, then $\beta_{ijk} = 0$ and otherwise β_{ijk} is the class of m_{ijk} in $\mathrm{gr}^{\mu_{ijk}}\mathscr{O}^\sim_{\eta(r)}$.

Claim 2.3.24. $\mathscr{C}(sr)/\mathfrak{m}\mathscr{C}(sr) \simeq \mathrm{gr}^\bullet\mathscr{B}(r)^\sim$ for every $s \in \Gamma_K^+ \setminus \{1\}$ sufficiently close to 1.

Proof of the claim. With the notation of (2.3.23) we have

$$\mathscr{C}(sr)/\mathfrak{m}\mathscr{C}(sr) = \mathscr{O}^\sim_{\eta(sr)}\left[\frac{c_1}{a^{\gamma_1}},\dots,\frac{c_d}{a^{\gamma_d}}\right].$$

We define an isomorphism

$$\mathrm{gr}^\bullet\mathscr{O}^\sim_{\eta(r)} \simeq \mathscr{O}^\sim_{\eta(sr)} \tag{2.3.25}$$

by the rule:

$$g \mapsto g/a^n \pmod{\mathfrak{m}\mathscr{O}^+_{\eta(sr)}} \qquad \text{for every } g \in \mathrm{gr}^n\mathscr{O}^\sim_{\eta(r)}.$$

Via (2.3.25), $\mathscr{C}(sr)/\mathfrak{m}\mathscr{C}(sr)$ becomes a free $\mathrm{gr}^\bullet\mathscr{O}^\sim_{\eta(r)}$-module, whose rank d is the same as the rank of $\mathrm{gr}^\bullet\mathscr{B}(r)^\sim$. Obviously, we would like to extend the isomorphism (2.3.25) by setting

$$\overline{c}_i \mapsto c_i/a^{\gamma_i} \qquad \text{for every } i = 1,\dots,d. \tag{2.3.26}$$

In view of (2.3.22), in order to prove that (2.3.26) yields a well-defined ring homomorphism, it suffices to show that the class of $m_{ijk}/a^{\gamma_i+\gamma_j-\gamma_k}$ in $\mathscr{O}^\sim_{\eta(sr)}$ agrees with the image of β_{ijk} under (2.3.25), whenever s is sufficiently close to 1. This can be checked easily by inspecting the definitions; we leave the details to the reader. ◊

Claim 2.3.27. $\mathscr{C}(sr) = \mathscr{B}(sr)^+$ whenever $s \in \Gamma_K^+$ is sufficiently close to 1.

Proof of the claim. In view of claim 2.3.24 and lemma 2.2.17(iv), it suffices to show that $\mathrm{gr}^\bullet\mathscr{B}(r)^\sim$ is reduced. But this is clear, since the norm $|\cdot|^\natural_{\mathrm{sp},\eta(r)}$ on $\mathscr{B}(r)^\sim$ is power-multiplicative. ◊

In view of claim 2.3.27 we have

$$\mathfrak{d}^+(sr) = \det\left(\mathrm{Tr}_{\mathscr{B}(rs)^+/\kappa(rs)^+}\left(\frac{c_i}{a^{\gamma_i}} \otimes \frac{c_j}{a^{\gamma_j}}\right) \mid 1 \leq i,j \leq d\right) = a^{-2\cdot\sum_i\gamma_i} \cdot \mathfrak{d}^+(r)_{\eta(rs)} \tag{2.3.28}$$

whenever $s := |a|$ is sufficiently close to 1 (here $\mathfrak{d}^+(r)_{\eta(rs)}$ denotes the image of $\mathfrak{d}^+(r)$ in $\kappa(rs)^+$; this is well-defined whenever s is sufficiently close to 1). However, one deduces easily from (2.3.10) that $|\mathfrak{d}^+(t)|^\flat_{\eta(t)} = |\mathfrak{d}^\flat(t)|_{\eta(t)^\flat}$ for every $t \in (a,b] \cap \Gamma_K$. Since we have as well:

$$|\mathscr{B}(r)^+|^\natural_{\mathrm{sp},\eta(r)} = \sum_{i=1}^d \gamma_i$$

the contention follows from lemma 2.2.20(ii). $\square$

2.3.29. Proposition 2.3.17 expresses the slope of the discriminant function at a given radius r as a local invariant depending only on the behaviour of the morphism f over the point $\eta(r)$. We wish now to consider a special situation, where the slope can also be obtained as a global invariant of the ring $\Gamma(X, \mathscr{O}^+_X)$. Namely, suppose that $g : X \to \mathbb{D}(1)$ is a finite, flat and generically étale morphism; proceeding as in the foregoing we attach to g a discriminant function δ_g, which clearly shall be defined over the set $[0, +\infty) \cap \log \Gamma_K = -\log \Gamma^+_K$. However, our present aim is to compute the right slope of $\delta_g(\rho)$ in a small neighborhood of $\rho = 0$. To this purpose, let $B^\circ := \Gamma(X, \mathscr{O}^+_X)$ and $B := B^\circ \otimes_{K^+} K$; according to proposition 2.3.5(i), B° is a free module, necessarily of rank $d := \deg(g)$, over the ring $A(1)^\circ = \Gamma(\mathbb{D}(1), \mathscr{O}^+_{\mathbb{D}(1)})$. Clearly $B \otimes_{A(1)} \kappa(1) = \mathscr{B}(1)$, hence the natural map

$$B^\circ_\eta := B^\circ \otimes_{A(1)^\circ} \kappa(1)^+ \to \mathscr{B}(1)^+ \tag{2.3.30}$$

is injective. Let $\mathfrak{d}^\circ_g := \mathfrak{d}_{B^\circ/A(1)^\circ}$; combining lemmata 2.1.11 and 1.2.26 we deduce:

$$|\mathfrak{d}^+_g(1)|^\natural_{\eta(1)} - 2 \cdot |\mathscr{B}(1)^+|^\natural_{\mathrm{sp},\eta(1)} = |\mathfrak{d}^\circ_g|^\natural_{\eta(1)} - 2 \cdot |B^\circ_\eta|^\natural_{\mathrm{sp},\eta(1)}. \tag{2.3.31}$$

Notice that the left-hand side of this identity calculates the right slope of δ_g at the point $\rho = 0$, hence the right-hand side is the sought global formula for this slope.

2.3.32. The contribution $|B^\circ_\eta|^\natural_{\mathrm{sp},\eta(1)}$ can be further analyzed. Indeed, let us set

$$B^{\circ h}_\eta := B^\circ \otimes_{A(1)^\circ} \kappa(1)^{\wedge h+}.$$

The ring $B^{\circ h}_\eta$ is henselian along the ideal $\mathfrak{p}_\eta B^{\circ h}_\eta$, where $\mathfrak{p}_\eta$ is the maximal ideal of $\kappa(1)^+$. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_k \subset B^\circ_\eta$ be the finitely many prime ideals lying over $\mathfrak{p}_\eta$; the ring $B^{\circ h}_\eta$ decomposes as a direct product of henselian local rings:

$$B^{\circ h}_\eta = B^{\circ h}_{\mathfrak{q}_1} \times \cdots \times B^{\circ h}_{\mathfrak{q}_k}.$$

For every $i = 1, \dots, k$ set

$$\mathfrak{F}(\mathfrak{q}_i) := \{x \in g^{-1}(\eta(1)) \mid \kappa(x)^+ \text{ dominates } B^\circ_{\mathfrak{q}_i}\}.$$

After completion, henselization and localization at $\mathfrak{q}_i$, the map (2.3.30) yields injective ring homomorphisms (see the proof of lemma 2.2.17(i)) :

$$B^{\circ h}_{\mathfrak{q}_i} \to \mathscr{B}(1)^{\wedge h+}_{\mathfrak{q}_i} \simeq \prod_{x \in \mathfrak{F}(\mathfrak{q}_i)} \kappa(x)^{\wedge h+} \tag{2.3.33}$$

More precisely, let $\overline{\kappa}(\mathfrak{q}_i)$ (resp. $\overline{\kappa}(x)$) be the residue field of $B^{\circ h}_{\mathfrak{q}_i}$ (resp. of $\kappa(x)^{\wedge h+}$); the maps (2.3.33) induce isomorphisms $\overline{\kappa}(\mathfrak{q}_i) \xrightarrow{\sim} \overline{\kappa}(x)$ for every $x \in \mathfrak{F}(\mathfrak{q}_i)$, hence the image of (2.3.33) lands in the *seminormalization* of $B^{\circ h}_{\mathfrak{q}_i}$, *i.e.* the subring

$$B^{h\nu}_{\mathfrak{q}_i} := \kappa(x_1)^{\wedge h+} \times_{\overline{\kappa}(\mathfrak{q}_i)} \cdots \times_{\overline{\kappa}(\mathfrak{q}_i)} \kappa(x_r)^{\wedge h+}$$

(the fibre product over $\overline{\kappa}(\mathfrak{q}_i)$ of the rings $\kappa(x_i)^{h+}$, where $\{x_1, \dots, x_r\} = \mathfrak{F}(\mathfrak{q}_i)$). Let us set:

$$\alpha(\mathfrak{q}_i) := |F_0(B^{h\nu}_{\mathfrak{q}_i}/B^{\circ h}_{\mathfrak{q}_i})|^{\wedge h \natural}_{\eta(1)} \qquad \text{for every } i = 1, \dots, k.$$

Lemma 2.3.34. *With the notation of* (2.3.32)*, we have:*

$$2 \cdot |B^\circ_\eta|^\natural_{\mathrm{sp},\eta(1)} = \deg(g) + \sum_{i=1}^k (2\alpha(\mathfrak{q}_i) + \sharp\mathfrak{F}(\mathfrak{q}_i) - 2).$$

Proof. (Here $\sharp\mathfrak{F}(\mathfrak{q}_i)$ denotes the cardinality of the finite set $\mathfrak{F}(\mathfrak{q}_i)$.) First of all, we remark that $\overline{\kappa}(\mathfrak{q}_i) \simeq \overline{\kappa}(\eta(1)) \simeq \kappa(1)^+/\xi\kappa(1)^+$, where $\xi \in A(1)$ is an element such that $|\xi|_{\eta(1)} = 1 - \varepsilon$, hence $|F_0(\overline{\kappa}(\mathfrak{q}_i))|^\natural_{\eta(1)} = 1$; it follows easily that

$$|F_0(\mathscr{B}(1)^{\wedge h+}_{\mathfrak{q}_i}/B^{h\nu}_{\mathfrak{q}_i})|^{\wedge h\natural}_{\eta(1)} = \sharp\mathfrak{F}(\mathfrak{q}_i) - 1 \qquad \text{for every } i = 1, \dots, k.$$

Hence

$$\begin{aligned} 2 \cdot |B^\circ_\eta|^\natural_{\mathrm{sp},\eta(1)} &= 2 \cdot |B^{\circ h}_\eta|^{\wedge h\natural}_{\mathrm{sp},\eta(1)} = 2 \cdot (|\mathscr{B}(1)^{\wedge h+}|^{\wedge h\natural}_{\mathrm{sp},\eta(1)} + |F_0(\mathscr{B}(1)^{\wedge h+}/B^{\circ h}_\eta)|^{\wedge h\natural}_{\eta(1)}) \\ &= 2 \cdot |\mathscr{B}(1)^+|^\natural_{\mathrm{sp},\eta(1)} + 2\textstyle\sum_{i=1}^k (|F_0(\mathscr{B}(1)^{\wedge h+}_{\mathfrak{q}_i}/B^{h\nu}_{\mathfrak{q}_i})|^{\wedge h\natural}_{\eta(1)} + |F_0(B^{h\nu}_{\mathfrak{q}_i}/B^{\circ h}_{\mathfrak{q}_i})|^{\wedge h\natural}_{\eta(1)}) \\ &= 2 \cdot |\mathscr{B}(1)^+|^\natural_{\mathrm{sp},\eta(1)} + 2\textstyle\sum_{i=1}^k (\sharp\mathfrak{F}(\mathfrak{q}_i) - 1 + \alpha(\mathfrak{q}_i)) \\ &= \deg(g) - \sharp g^{-1}(\eta(1)) + 2\textstyle\sum_{i=1}^k (\sharp\mathfrak{F}(\mathfrak{q}_i) - 1 + \alpha(\mathfrak{q}_i)) \end{aligned}$$

where the last equality holds by lemma 2.2.17(iii). Since clearly

$$\sum_{i=1}^k \sharp\mathfrak{F}(\mathfrak{q}_i) = \sharp g^{-1}(\eta(1))$$

the assertion follows. □

Theorem 2.3.35. *With the notation of* (2.3.12) *:*

(i) *δ_f extends to a continuous, piecewise linear function $\delta : [\log 1/b, \log 1/a] \to \mathbb{R}_{\geq 0}$ with integer slopes.*
(ii) *If moreover f is étale, then δ is convex.*

Proof. Let $(F, |\cdot|_F)$ be an algebraically closed valued field extension of K with $|F|_F = \mathbb{R}_{\geq 0}$, and denote by $f_F : X \times_{\mathrm{Spa}\, K} \mathrm{Spa}\, F \to \mathbb{D}(a, b) \times_{\mathrm{Spa}\, K} \mathrm{Spa}\, F$ the morphism deduced by base change of f; using proposition 2.3.5(iii) one sees that δ_{f_F} agrees with δ_f wherever the latter is defined. Hence we can and do assume from start that $|K| = \mathbb{R}_{\geq 0}$. Now, for the proof of (i) it suffices to show that δ_f is piecewise linear in the neghborhood of every real number $\rho := \log 1/r \in [\log 1/b, \log 1/a]$. Using a morphism g as in example 2.3.16, one reduces to consider the case where $r > a$, and study the function δ_f in a small interval $[\rho, \rho + x]$. In such situation, the more precise proposition 2.3.17 shows that the assertion holds.

Suppose next that f is étale. In order to show (ii), we need to study the function δ in any small neighborhood of the form $[\log 1/r - x, \log 1/r + x] \subset [\log 1/b, \log 1/a]$. Assertion (ii) then means that the function $\rho \mapsto \delta(\log 1/r - \rho) + \delta(\log 1/r + \rho)$ has positive derivative in a neighborhood of 0. We can assume that $r = 1$ and $x = -\log|\pi|$ for some $\pi \in \mathfrak{m}$, so we reduce to consider an étale morphism $f : X \to \mathbb{D}(a, a^{-1})$ (for $a := |\pi|$). In view of (2.3.14) we can further reduce to studying the morphism $h := \mathrm{Spa}(\psi_K) \circ f : X \to \mathbb{D}(1)$, defined as in example 2.3.13, and then we have to show that the left slope of δ_h is negative in a small neighborhood $(x, 0]$. Say that $X = \mathrm{Spa}\, B$; in the notation of (2.3.31) we have

$$|\mathfrak{d}^\circ_h|^\natural_{\eta(1)} = |(\mathfrak{d}_\psi)^d|^\natural_{\eta(1)} = 2d = \deg(g) \tag{2.3.36}$$

by example 2.1.12 and proposition 2.1.4. Finally, in view of (2.3.31), (2.3.36), proposition 2.3.17 and lemma 2.3.34, the sought assertion is implied by the following:

Claim 2.3.37. Resume the notation of (2.3.32). Then $\sharp\mathfrak{F}(\mathfrak{q}_i) \geq 2$ for every $i = 1, \dots, k$.

Proof of the claim. Recall that $\mathfrak{q}_1, \dots, \mathfrak{q}_k$ are by definition the prime ideals of B°_η lying over the maximal ideal of $\kappa(1)^{h+}$, or what is the same, the prime ideals of B° lying over the maximal ideal $\mathfrak{p} := \mathfrak{m}A(1)^\circ + \xi A(1)^\circ$ of $A(1)^\circ$. Now, we have already observed (example 2.3.13) that $A(a, a^{-1})^\circ \simeq K^+\langle S, T\rangle/(ST - \pi^2)$, and ψ is the map $K^+\langle\xi\rangle \to K^+\langle S, T\rangle/(ST - \pi^2)$ such that $\xi \mapsto S + T$. Hence $A(a, a^{-1})^\circ/\mathfrak{p}A(a, a^{-1})^\circ \simeq K^\sim[S, T]/(ST, S + T) \simeq K^\sim[S]/(S^2)$. Thus, there is exactly one prime ideal $\mathfrak{P} \subset A(a, a^{-1})^\circ$ lying over $\mathfrak{p}$ and necessarily $\mathfrak{q}_i \cap A(a, a^{-1})^\circ = \mathfrak{P}$ for every $i = 1, \dots, k$. On the other hand, the fibre $\psi^{-1}(\eta(1)) \subset \mathbb{D}(a, a^{-1})$ consists of the two valuations $\eta'(a), \eta(1/a)$, and clearly both of them dominate the local ring $A(a, a^{-1})^\circ_{\mathfrak{P}}$. It is now a standard fact that, for each prime ideal $\mathfrak{q}_i$, there are valuations η_1, η_2 on B which extend respectively $\eta'(a)$ and $\eta(1/a)$, and which dominate the local ring $B^\circ_{\mathfrak{q}_i}$. By lemma 2.2.12 we have $\eta_1, \eta_2 \in \operatorname{Spa} B$, whence the claim. □

Remark 2.3.38. The continuity and piecewise linearity of the function δ_f are also proved in the preprint [45] of T.Schmechta. He also obtains some interesting results in the case where the base field has positive characteristic. His methods are refinements of those of Lütkebohmert [40]. (However, as far as I understand, he does not prove the convexity of the function δ_f.)

2.4. **The p-adic Riemann existence theorem.** In this section we show how to use theorem 2.3.35 to solve the so-called p-adic Riemann existence problem in case K is a field of characteristic zero. We choose an argument that maximizes the use of valuation theory; see remark 2.4.8 for some indications of an alternative, slightly different proof.

2.4.1. Recall that a finite étale covering $f : X \to \mathbb{D}(a, b)$ is said to be *of Kummer type*, if there exists an integer $n > 0$ and an isomorphism $g : \mathbb{D}(a^{1/n}, b^{1/n}) \xrightarrow{\sim} X$ such that $f \circ g = \operatorname{Spa}\phi$, where $\phi : A(a, b) \to A(a^{1/n}, b^{1/n})$ is the map of K-affinoid algebras given by the rule $\xi \mapsto \xi^n$ (notation of (2.2.7)).

Lemma 2.4.2. *Let $f : X \to \mathbb{D}(a, b)$ and $g : Y \to X$ be two finite étale coverings. Then f and g are of Kummer type, if and only if the same holds for $f \circ g$.*

Proof. Left to the reader. □

Theorem 2.4.3. *Suppose that K is algebraically closed of characteristic zero, and let $f : X \to \mathbb{D}(a, b)$ be a finite étale morphism of degree d. There is a constant $c := c(d) \in (0, 1]$ such that the restriction $f^{-1}(\mathbb{D}(c^{-1}a, cb)) \to \mathbb{D}(c^{-1}a, cb)$ splits as the disjoint union of finitely many finite coverings of Kummer type.*

Proof. First of all, let $f' : Y \to \mathbb{D}(a, b)$ be the smallest Galois étale covering that dominates f (*i.e.* such that f' factors through f); it is well-known that the degree of f' is bounded by $d!$. Suppose now that the theorem is known for f'; then we may find $c \in (0, 1]$ such that the restriction of f' to the preimage of $\mathbb{D}(c^{-1}a, cb)$ is of Kummer type. Using lemma 2.4.2 we deduce that the same holds for the restriction of f to $f^{-1}(\mathbb{D}(c^{-1}a, cb))$. Hence, we may replace f by f' and assume from start that f is a Galois covering, say of finite group G.

Next, we consider the function $\delta : [\log 1/b, \log 1/a] \cap \log \Gamma_K \to \mathbb{R}_{\geq 0}$ corresponding to the covering f. To start out, lemma 2.1.10 implies that δ admits an upper bound that depends only on d; since δ is convex, piecewise linear and non-negative and since its slopes are integers (theorem 2.3.35), it follows easily that we may find a constant $c \in (0, 1]$, depending only on the degree d, such that δ is linear (indeed constant) on the interval $[\log 1/(bc), \log c/a] \cap \log \Gamma_K$. We may therefore assume from start that δ is linear. Also, we may assume that $a < 1$ and $b = a^{-1}$, in which case we let $g := \operatorname{Spa}\psi_K : \mathbb{D}(a, a^{-1}) \to \mathbb{D}(1)$, where ψ is defined as in example 2.3.13, and $h := g \circ f : X \to \mathbb{D}(1)$. Let $\mathfrak{p} := \mathfrak{m}A(1)^\circ + \xi A(1)^\circ \subset A(1)^\circ$; in the proof of claim 2.3.37 we have established that there exists a unique prime ideal $\mathfrak{P} \subset A(a, a^{-1})^\circ$

lying over $\mathfrak{p}$, and both rings $\kappa(\eta'(a))^+$ and $\kappa(b)^+$ dominate the localization $A(a,b)^\circ_{\mathfrak{P}}$; denote also $\mathfrak{q}_1,\dots,\mathfrak{q}_k \subset B^\circ$ the finitely many prime ideals lying over $\mathfrak{p}$.

The natural map $A(1)^\circ \to A(1)^{\circ\wedge} \simeq K^+[[\xi]]$ from $A(1)^\circ$ to its ξ-adic completion, factors through the henselization $A(1)^{\circ h}_{\mathfrak{p}}$ of $A(1)^\circ$ along its ideal $\mathfrak{p}$; hence $B^{\circ\wedge} := B^\circ \otimes_{A(1)^\circ} A(1)^{\circ\wedge}$ decomposes as a direct product of algebras : $B^{\circ\wedge} \simeq C_1 \times \cdots \times C_k$. Moreover, for every $t \in \mathfrak{m}\setminus\{0\}$, the map $A(1)^\circ \to A(|t|)^\circ$ induced by the open imbedding $\mathbb{D}(|t|) \to \mathbb{D}(1)$ factors through $A(1)^{\circ\wedge}$, and induces an isomorphism $A(1)^\circ/\mathfrak{p} \xrightarrow{\sim} A(|t|)^\circ/\mathfrak{p}_t$, where $\mathfrak{p}_t := \mathfrak{m}A(|t|)^\circ + (\xi/t)A(|t|)^\circ$. It follows that the prime ideals of

$$B^\circ \otimes_{A(1)^\circ} A(|t|)^\circ \simeq (C_1 \otimes_{A(1)^{\circ\wedge}} A(|t|)^\circ) \times \cdots \times (C_k \otimes_{A(1)^{\circ\wedge}} A(|t|)^\circ)$$

lying over $\mathfrak{p}_t$ are in natural bijection with the prime ideals $\mathfrak{q}_1,\dots,\mathfrak{q}_k$, and therefore every factor $C_i(t) := C_i \otimes_{A(1)^\circ} A(|t|)^\circ$ contains exactly one of these prime ideals. Notice that $g^{-1}(\mathbb{D}(|t|)) = \mathbb{D}(a/|t|,|t|/a)$, whence natural isomorphisms :

$$f^{-1}(\mathbb{D}(a/|t|,|t|/a)) \simeq \mathrm{Spa}(C_1(t)\otimes_{K^+} K) \amalg \cdots \amalg \mathrm{Spa}(C_k(t)\otimes_{K^+} K)$$

so that the restriction of δ to $[\log a/|t|, \log|t|/a] \cap \log\Gamma_K$ decomposes as a sum $\delta = \delta_1 + \cdots + \delta_k$, where δ_i is the discriminant function of the restriction

$$f_i : \mathrm{Spa}(C_i(t)\otimes_{K^+} K) \to \mathbb{D}(a/|t|,|t|/a)$$

for every $i \leq k$. Since every such δ_i is still convex, and their sum is linear, it follows that δ_i is linear for every $i \leq k$. We remark as well, that each f_i is still a Galois covering, whose Galois group is the subgroup of G that stabilizes $\mathfrak{q}_i$, for the natural action of G on the set $\{\mathfrak{q}_1,\dots,\mathfrak{q}_k\}$. Clearly it suffices to prove the theorem separately for every étale covering f_i, hence we may replace from start f by f_i, and assume additionally that $k = 1$, in which case we shall write $\mathfrak{q}$ instead of $\mathfrak{q}_1$. Define $\alpha(\mathfrak{q})$, $\mathfrak{F}(\mathfrak{q})$ as in (2.3.32). By inspection of the proof of theorem 2.3.35 we deduce that δ is linear precisely when :

$$\alpha(\mathfrak{q}) = 0 \quad \text{and} \quad \sharp\mathfrak{F}(\mathfrak{q}) = 2. \tag{2.4.4}$$

Especially, the preimages $f^{-1}(\eta'(a))$ and $f^{-1}(\eta(b))$ both consist of precisely one point; let $x \in X$ be the only point lying over $\eta(b)$. We deduce a Galois extension of valued fields :

$$\kappa(b) \to \kappa(x)$$

whose Galois group is isomorphic to G. Since the residue field of K is algebraically closed, the residue field extension $\overline{\kappa}(\eta(b)) \to \overline{\kappa}(x)$ is trivial, and therefore G is a solvable group. Thus, we may factor f as a composition of finitely many étale coverings :

$$X_n := X \xrightarrow{g_n} X_{n-1} \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_1} X_0 := \mathbb{D}(a,b)$$

such that the degree of g_i is a prime number for every $i \leq n$. Using lemma 2.4.2 and an easy induction, we may then further reduce to the case where G is a cyclic group of prime order d.

Such coverings are classified by the étale cohomology group $H := H^1(\mathbb{D}(a,b)_{\text{ét}}, \mathbb{Z}/d\mathbb{Z})$ (where $\mathbb{D}(a,b)_{\text{ét}}$ denotes the étale site of $\mathbb{D}(a,b)$, as defined in [30]). The latter can be computed by the Kummer exact sequence (on the étale site of $\mathbb{D}(a,b)$) :

$$0 \to \mu_d \to \mathscr{O}^\times \xrightarrow{(-)^d} \mathscr{O}^\times \to 0$$

(recall that K has characteristic zero) and since the Picard group of $\mathbb{D}(a,b)$ is trivial ([24, Th.2.2.9(3)]), one obtains a natural isomorphism :

$$H \xrightarrow{\sim} A(a,b)^\times/(A(a,b)^\times)^d$$

where $A(a,b)^\times$ denotes the invertible elements of $A(a,b)$. Under this isomorphism, the Kummer coverings of degree d correspond to the equivalence classes of the sections ξ^j, for $j = 0,\dots,d-1$ (notation of (2.2.7)). Thus, we come down to verifying the following :

Claim 2.4.5. There exists a constant $c := c(d) \in (0,1]$ such that, for every $u \in A(a,b)^\times$, the restriction $u' := u_{|\mathbb{D}(a/c,bc)}$ can be written in the form : $u' = v^d \cdot \xi^j$ for some $v \in A(a/c,cb)^\times$ and $0 \leq j \leq d-1$.

Proof of the claim. Let $\alpha, \beta \in K^\times$ such that $|\alpha| = a$, $|\beta| = b$; it is well known that every invertible element u of $A(a,b)$ can be written in the form $u = \gamma \cdot \xi^n \cdot (1+h)$ where $\gamma \in K^\times$, $n \in \mathbb{Z}$ and $h \in A(a,b)^\circ$ of the form

$$h(\xi) = \sum_{k \in \mathbb{Z} \setminus \{0\}} h_k \xi^k \in K^+\langle \xi/\alpha, \beta/\xi \rangle \tag{2.4.6}$$

with $|h|_{\sup} < 1$. Hence we are reduced to showing that $1+h$ admits a d-th root, after restriction to a smaller annulus $\mathbb{D}(a/c, b/c)$. This is clear in case $d \neq p$, in which case we may even choose $c = 1$. Finally, suppose that $d = p$; it is well known that $1+h$ admits a p-th root as soon as

$$|h|_{\sup} < |p|^{1/(p-1)}. \tag{2.4.7}$$

Using the explicit description (2.4.6) we may easily determine $c \in (0,1]$ such that the estimate (2.4.7) holds for the restriction $h_{|\mathbb{D}(a/c,cb)}$. □

Remark 2.4.8. Keep the notation of the proof of theorem 2.4.3. Alternatively, one may deduce from (2.4.4) that $\mathfrak{q}$ is an ordinary double point of the analytic reduction of X, in which case [7, Prop.2.3] shows that the corresponding formal fibre is an open annulus, and then theorem 2.4.3 follows without too much trouble.

3. Study of the conductors

3.1. **Algebraization.** Let V be a henselian local ring, s the closed point of $\operatorname{Spec} V$ and $\kappa(s)$ its residue field. For every affine V-scheme X we let $X_s := X \times_{\operatorname{Spec} V} \operatorname{Spec} \kappa(s)$. More generally, let $X := \operatorname{Spec} R$ be any affine scheme, and $Z \subset X$ a closed subscheme, say $Z = V(I)$ for an ideal $I \subset R$; we denote by R^h_I the henselization of R along the ideal I and by $R^\wedge_I$ the I-adic completion of R. The *henselization* of X along Z is the affine scheme $X^h_{/Z} := \operatorname{Spec} R^h_I$.

Lemma 3.1.1. *Let A be a noetherian ring, $I, J \subset A$ two ideals.*

(i) *The natural commutative diagram*

$$\begin{tikzcd} A^\wedge_{I\cap J} \arrow[r] \arrow[d] & A^\wedge_I \arrow[d] \\ A^\wedge_J \arrow[r] & A^\wedge_{I+J} \end{tikzcd}$$

is cartesian.

(ii) *Moreover, for every $n \in \mathbb{N}$, set $A_n := A/J^n$. Then there is a natural isomorphism of A-algebras:*

$$A^\wedge_{I+J} \xrightarrow{\sim} \lim_{n\in\mathbb{N}} A^\wedge_{n,I}.$$

Proof. (i): Using [19, Ch.0, Lemme 19.3.10.2] we see that the natural commutative diagram

$$\begin{tikzcd} A/(I^n \cap J^n) \arrow[r] \arrow[d] & A/I^n \arrow[d] \\ A/J^n \arrow[r] & A/(I^n + J^n) \end{tikzcd}$$

is cartesian for every $n > 0$. Set $A' := \lim_{n\in\mathbb{N}} A/(I^n + J^n)$ and $A'' := \lim_{n\in\mathbb{N}} A/(I^n \cap J^n)$. We deduce easily a cartesian commutative diagram:

$$\begin{array}{ccc} A'' & \longrightarrow & A^\wedge_I \\ \downarrow & & \downarrow \\ A^\wedge_J & \longrightarrow & A' \end{array}$$

and it remains only to show that the natural maps $A' \to A^\wedge_{I+J}$ and $A^\wedge_{I\cap J} \to A''$ are isomorphisms. For the former, it suffices to remark that

(3.1.2) $$(I + J)^{2n-1} \subset I^n + J^n \subset (I + J)^n$$

for every $n > 0$. For the latter, one uses the Artin-Rees lemma [41, Th.8.5] to show that for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$I^m \cap J^n \subset I^n J^n \subset (I \cap J)^n$$

from which (i) follows easily. (ii) is an easy consequence of (3.1.2); we leave the details to the reader. □

Theorem 3.1.3. *In the situation of* (3.1)*, suppose that $\kappa(s)$ is a perfect field. Let X be an affine finitely presented V-scheme of pure relative dimension one, $\Sigma \subset X_s$ a finite subset such that $X_s \setminus \Sigma$ is smooth over $\operatorname{Spec}\kappa(s)$. Then there exists a projective V-scheme Y of pure relative dimension one and an open affine subset $U \subset Y$ with an isomorphism of V-schemes*

(3.1.4) $$X^h_{/X_s} \simeq U^h_{/U_s}.$$

Moreover, U_s is dense in Y_s and Y is smooth over $\operatorname{Spec} V$ at all the points of $Y_s \setminus U_s$.

Proof. We begin with the following:

Claim 3.1.5. There exists a projective purely one-dimensional $\kappa(s)$-scheme Y_0 and a dense open imbedding of $\kappa(s)$-schemes

(3.1.6) $$X_s \subset Y_0$$

such that Y_0 is smooth over $\operatorname{Spec}\kappa(s)$ at all the points of $Y_0 \setminus X_s$.

Proof of the claim. This is standard : one picks a projective $\kappa(s)$-scheme $\overline{X}_s$ containing X_s as a dense open subscheme, and let X'_s be the normalization of $\overline{X}_s \setminus \Sigma$. By [9, Ch.V, §3.2, Th.2] we know that X'_s is of finite type over $\kappa(s)$, and since X'_s has dimension one, we know as well that all its local rings are regular, hence they are formally smooth over $\kappa(s)$, in view of [41, §28, Lemma 1] and [22, Ch.IV, Prop.17.5.3]. One can then glue X_s and X'_s along their common open subscheme $X_s \setminus \Sigma$; the resulting scheme Y_0 will do. ◊

We can write as usual V as the colimit of a filtered family $(V_\lambda \mid \lambda \in \Lambda)$ of noetherian local subrings of V, essentially of finite type over an excellent discrete valuation ring, and such that the inclusion maps $j_\lambda : V_\lambda \to V$ are local ring homomorphisms. Then j_λ extends to a map $V^h_\lambda \to V$ from the henselization of V_λ, and V is still the colimit of the filtered family $(V^h_\lambda \mid \lambda \in \Lambda)$. For some $\lambda \in \Lambda$ we can find an affine finitely presented V^h_λ-scheme X_λ and an isomorphism of V-schemes:

$$\operatorname{Spec} V \times_{\operatorname{Spec} V^h_\lambda} X_\lambda \xrightarrow{\sim} X.$$

For every $\lambda \in \Lambda$, let k_λ be the residue field of V^h_λ; we can even choose λ in such a way that the scheme Y_0 provided by claim 3.1.5 descends to a projective k_λ-scheme $Y_{0,\lambda}$, so that $Y_0 \simeq \operatorname{Spec}\kappa(s) \times_{\operatorname{Spec} k_\lambda} Y_{0,\lambda}$. Let $\Sigma_\lambda \subset Y_{0,\lambda}$ be the image of Σ; we can furthermore assume that $Y_{0,\lambda}$ is smooth over $\operatorname{Spec} k_\lambda$ outside Σ_λ, and that there exists an open imbedding of k_λ-schemes

(3.1.7) $$\operatorname{Spec} k_\lambda \times_{\operatorname{Spec} V^h_\lambda} X_\lambda \subset Y_{0,\lambda}$$

inducing (3.1.6), after base change to $\mathrm{Spec}\,\kappa(s)$ (see [22, Ch.IV, Prop.17.7.8]). At the cost of trading the residue field $\kappa(s)$ with a non-perfect field, we can then replace the given ring V by one such V^h_λ, the scheme X by X_λ and Σ by Σ_λ; hence we can assume that V is the henselization of a ring of essentially finite type over an excellent discrete valuation ring, and additionally, that there exists a projective $\kappa(s)$-scheme Y_0 as in claim 3.1.5.

Let $\mathfrak{n} \subset V$ be the maximal ideal; denote by V-$\mathbf{Alg}$ the category of V-algebras, by $\mathbf{Set}$ the category of sets. We define a functor $\mathscr{F} : V\text{-}\mathbf{Alg} \to \mathbf{Set}$ as follows. For a V-algebra A, $\mathscr{F}(A)$ is the set of equivalence classes of data of the form $(Z_A, Y_A, f_A, g_A, h_A)$ where:

- Z_A and Y_A are finitely presented A-schemes, and Y_A is projective over $\mathrm{Spec}\,A$.
- $f_A : Z_A \to X_A := \mathrm{Spec}\,A \times_{\mathrm{Spec}\,V} X$ and $g_A : Z_A \to Y_A$ are étale morphisms of A-schemes.
- $h_A : Y_{A,s} := \mathrm{Spec}\,A/\mathfrak{n}A \times_{\mathrm{Spec}\,A} Y_A \to \mathrm{Spec}\,A/\mathfrak{n}A \times_{\mathrm{Spec}\,\kappa(s)} Y_0$ is an isomorphism.
- The restriction $f_{A,s} : Z_{A,s} := \mathrm{Spec}\,A/\mathfrak{n}A \times_{\mathrm{Spec}\,A} Z_A \to X_{A,s} := \mathrm{Spec}\,A/\mathfrak{n}A \times_{\mathrm{Spec}\,V} X$ is an isomorphism.
- The restriction $g_{A,s} : Z_{A,s} \to Y_{A,s}$ is an open imbedding and the morphism
$$\mathrm{Spec}\,A/\mathfrak{n}A \times_{\mathrm{Spec}\,\kappa(s)} (3.1.6) : X_{A,s} \to Y_{A,s}$$
agrees with $h_A \circ g_{A,s} \circ f^{-1}_{A,s}$.
- Y_A is smooth over $\mathrm{Spec}\,A$ at all the points of $Y_{A,s} \setminus g_{A,s}(Z_{A,s})$.

Two data $(Z_A, Y_A, f_A, g_A, h_A)$ and $(Z'_A, Y'_A, f'_A, g'_A, h'_A)$ are said to be equivalent if there are isomorphisms of A-schemes $Z_A \xrightarrow{\sim} Z'_A$, $Y_A \xrightarrow{\sim} Y'_A$ such that the obvious diagrams commute. A map $A \to A'$ of V-algebras induces an obvious base change map $\mathscr{F}(A) \to \mathscr{F}(A')$.

Claim 3.1.8. $\mathscr{F}(V) \neq \varnothing$ if and only if there exists a pair (U, Y) as in the theorem, with an isomorphism of V-schemes $Y_s \xrightarrow{\sim} Y_0$.

Proof of the claim. Indeed, suppose we have found $(Z, Y, f, g, h) \in \mathscr{F}(V)$. Then we can set $U := g_V(Z) \subset Y$; the morphism f and g induce isomorphisms $f^h : Z^h_{/Z_s} \xrightarrow{\sim} X^h_{/X_s}$ and $g^h : Z^h_{/Z_s} \xrightarrow{\sim} U^h_{/X_s}$, and therefore the pair (U, Y) fulfills the conditions of the theorem; furthermore h yields an isomorphism $Y_s \xrightarrow{\sim} Y_0$. Conversely, suppose that a pair (U, Y) has been found that fulfills the conditions of the theorem; especially, (3.1.4) induces a morphism of V-schemes $g : X^h_{/X_s} \to U$. By [43, Ch.XI, §2, Th.2] $X^h_{/X_s}$ is the projective limit of a cofiltered family of morphisms of V-schemes $(f_\alpha : X_\alpha \to X \mid \alpha \in I)$ such that every X_α is affine and finitely presented over $\mathrm{Spec}\,V$, the induced morphisms $X_{\alpha,s} \to X_s$ are isomorphisms, and for every $x \in X_\alpha$ the induced map $\mathscr{O}_{X,f_\alpha(x)} \to \mathscr{O}_{X_\alpha,x}$ is local ind-étale. By [21, Ch.IV, Th.11.1.1] we deduce that there is an open (quasi-compact) subset Z_α of X_α containing $X_{\alpha,s}$ such that the restriction of f_α to Z_α is flat; then, up to shrinking Z_α, we can achieve that the morphism $Z_\alpha \to X$ is an étale neighborhood of X_s, and X is still the projective limit of the V-schemes Z_α. We can then find $\alpha \in I$ such that g factors through a morphism $g_\alpha : Z_\alpha \to U$; after further shrinking Z_α, the morphism g_α becomes étale. Hence, the datum $(Z_\alpha, Y, f_\alpha, g_\alpha)$ represents an element of $\mathscr{F}(V)$. $\Diamond$

Next, in view of [21, Ch.IV, Th.8.10.5] and [22, Ch.IV, Prop.17.7.8(ii)] one sees easily that $\mathscr{F}$ is a functor of finite presentation. Let $V^\wedge$ be the $\mathfrak{n}$-adic completion of V; according to Artin's approximation theorem [2, Th.I.12], every element of $\mathscr{F}(V^\wedge)$ can be approximated arbitrarily closely in the $\mathfrak{n}$-adic topology by elements of $\mathscr{F}(V)$, especially:

$$\mathscr{F}(V) \neq \varnothing \Leftrightarrow \mathscr{F}(V^\wedge) \neq \varnothing. \tag{3.1.9}$$

So finally, in view of (3.1.9) and claim 3.1.8, we can replace V by $V^\wedge$ and suppose from start that V is a complete noetherian local ring. Let $V_n := V/\mathfrak{n}^{n+1}$ and $X_n := \mathrm{Spec}\,V_n \times_{\mathrm{Spec}\,V} X$ for

every $n \in \mathbb{N}$. We endow V with its $\mathfrak{n}$-adic topology; the family $(X_n \mid n \in \mathbb{N})$ defines a unique affine formal $\mathrm{Spf}\, V$-scheme $\mathfrak{X}$.

Claim 3.1.10. There exists a proper $\mathrm{Spf}\, V$-scheme $\mathfrak{Y}$ with an open imbedding $\mathfrak{X} \subset \mathfrak{Y}$, such that the reduced fibre of $\mathfrak{Y}$ is V_0-isomorphic to Y_0, and such that $\mathfrak{Y}$ is formally smooth over $\mathrm{Spf}\, V$ at all points of $Y_0 \setminus X_0$.

Proof of the claim. In view of [18, Ch.III, §3.4.1] and [16, Ch.I, Prop.10.13.1], it suffices to lift the V_0-scheme Y_0 to a compatible family $(Y_n \mid n \in \mathbb{N})$ of schemes such that

- (i) for every $n \in \mathbb{N}$ there are isomorphisms of V_n-schemes: $Y_n \xrightarrow{\sim} \mathrm{Spec}\, V_n \times_{\mathrm{Spec}\, V_{n+1}} Y_{n+1}$;
- (ii) moreover, (3.1.6) lifts to a system of open imbeddings of V_n-schemes $X_n \subset Y_n$ compatible with the isomorphisms (ii);
- (iii) Y_n is smooth over $\mathrm{Spec}\, V_n$ at all points of $Y_n \setminus X_n$.

To this aim, we may assume that $\Sigma \neq \varnothing$, in which case $Y'_0 := Y_0 \setminus \Sigma$ is smooth and affine over $\mathrm{Spec}\, V_0$. One can then lift Y'_0 to a compatible system of schemes $(Y'_n \mid n \in \mathbb{N})$ satisfying a condition as the foregoing (i), and such that furthermore, Y'_n is smooth over $\mathrm{Spec}\, V_n$ for every $n \in \mathbb{N}$. Again some basic deformation theory shows that the imbedding $X_0 \setminus \Sigma \subset Y'_0$ lifts to a compatible system of open imbeddings $X_n \setminus \Sigma \subset Y'_n$ for every $n \in \mathbb{N}$. Therefore one can glue X_n and Y'_n along their common open subscheme $X_n \setminus \Sigma$; the resulting schemes Y_n will do. $\diamondsuit$

Next we are going to construct an invertible $\mathscr{O}_{\mathfrak{Y}}$-module on $\mathfrak{Y}$. To this aim we proceed as follows. Let $\{y_1, \dots, y_n\} := Y_0 \setminus X_0$. By construction $\mathfrak{Y}$ is formally smooth over $\mathrm{Spf}\, V$ at the point y_i, for every $i = 1, \dots, n$. For every $i \leq n$, the maximal ideal of $\mathscr{O}_{Y_0,y_i}$ is principal, say generated by the regular element $\bar{t}_i \in \mathscr{O}_{Y_0,y_i}$. The natural ring homomorphism $\mathscr{O}_{\mathfrak{Y},y_i} \to \mathscr{O}_{Y_0,y_i}$ is surjective, hence we can lift $\bar{t}_i$ to an element $t_i \in \mathscr{O}_{\mathfrak{Y},y_i}$.

Claim 3.1.11. t_i is a regular element in $\mathscr{O}_{\mathfrak{Y},y_i}$ for every $i = 1, \dots, n$.

Proof of the claim. By [19, Ch.0, Th.19.7.1] the ring $\mathscr{O}_{\mathfrak{Y},y_i}$ is flat over V; then the claim follows from [19, Ch.0, Prop.15.1.16]. $\diamondsuit$

In view of claim 3.1.11 we can find an open affine subscheme $\mathfrak{U}_i \subset \mathfrak{Y}$ such that $y_i \in \mathfrak{U}_i$ and t_i extends to a regular element of $\Gamma(\mathfrak{U}_i, \mathscr{O}_{\mathfrak{Y}})$. Finally, set $\mathfrak{V}_i := \mathfrak{Y} \setminus \{y_i\}$ and $\mathfrak{W}_i := \mathfrak{U}_i \cap \mathfrak{V}_i$; we define the invertible $\mathscr{O}_{\mathfrak{Y}}$-module $\mathscr{L}_i^\wedge$ by gluing the sheaves $\mathscr{O}_{\mathfrak{U}_i}$ (defined on $\mathfrak{U}_i$) and $\mathscr{O}_{\mathfrak{V}_i}$ (defined on $\mathfrak{V}_i$); the gluing map is the isomorphism

$$\mathscr{O}_{\mathfrak{V}_i|\mathfrak{W}_i} \xrightarrow{\sim} \mathscr{O}_{\mathfrak{U}_i|\mathfrak{W}_i} \qquad f \mapsto t_i \cdot f.$$

So, the global sections of $\mathscr{L}_i^\wedge$ are identified naturally with the pairs (f, g) where $f \in \Gamma(\mathfrak{U}_i, \mathscr{O}_{\mathfrak{Y}})$, $g \in \Gamma(\mathfrak{V}_i, \mathscr{O}_{\mathfrak{Y}})$ and $f_{|\mathfrak{W}_i} = t_i \cdot g_{|\mathfrak{W}_i}$. Clearly, $\mathscr{L}_i^\wedge/\mathfrak{n}\mathscr{L}_i^\wedge$ is the invertible sheaf $\mathscr{O}_{Y_0}(y_i)$ on Y_0; set $\mathscr{L}^\wedge := \mathscr{L}_1^\wedge \otimes_{\mathscr{O}_{\mathfrak{Y}}} \cdots \otimes_{\mathscr{O}_{\mathfrak{Y}}} \mathscr{L}_n^\wedge$. notice that, since X is affine, every irreducible component of X_0 meets $\{y_1, \dots, y_n\}$; it then follows from [39, §7.5, Prop.5] that $\mathscr{L}^\wedge/\mathfrak{n}\mathscr{L}^\wedge$ is ample on Y_0, and then [18, Ch.III, Th.5.4.5] shows that $\mathfrak{Y}$ is algebraizable to a projective scheme Y over $\mathrm{Spec}\, V$, and $\mathscr{L}^\wedge$ is the formal completion of an ample invertible $\mathscr{O}_Y$-module that we shall denote by $\mathscr{L}$. Especially, $\mathscr{L}/\mathfrak{n}\mathscr{L}$ is the ample sheaf $\mathscr{O}_{Y_0}(y_1 + \cdots + y_n)$ on Y_0.

By inspecting the construction, we see that $\mathscr{L} \simeq \mathscr{O}_Y(D)$, where D is an ample divisor on Y whose support $\mathrm{Supp}(D) \subset Y$ intersects Y_0 in precisely the closed subset $Y_0 \setminus X_0$. Therefore the open subset $U := Y \setminus \mathrm{Supp}(D)$ is affine and clearly $U \cap Y_0 = X_0$. Let $\mathfrak{U} \subset \mathfrak{Y}$ be the formal completion of U along its closed subscheme X_0; it follows that the imbedding $\mathfrak{X} \subset \mathfrak{Y}$ induces an isomorphism of affine formal $\mathrm{Spf}\, V$-schemes:

$$\mathfrak{U} \xrightarrow{\sim} \mathfrak{X}. \tag{3.1.12}$$

Let $Y^{\mathrm{sm}} \subset Y$ be the set of all points $x \in Y$ such that Y is smooth over $\mathrm{Spec}\, V$ at x; according to [20, Ch.IV, Cor.6.8.7] and [22, Ch.IV, Cor.17.5.2], Y^{sm} is an open subset of Y.

Claim 3.1.13. $Y_0 \setminus X_0 \subset Y^{\mathrm{sm}}$.

Proof of the claim. Indeed, for every $y \in Y_0$, the completions of the local rings $\mathscr{O}_{\mathfrak{Y},y}$ and $\mathscr{O}_{Y,y}$ are isomorphic topological V-algebras, therefore the claim follows from [19, Ch.IV, Prop.19.3.6] and [22, Ch.IV, Prop.17.5.3]. ◊

Claim 3.1.14. The (reduced) closed subscheme $Y^{\mathrm{sing}} := Y \setminus Y^{\mathrm{sm}}$ is finite over $\operatorname{Spec} V$, and moreover $Y^{\mathrm{sing}} \subset U$.

Proof of the claim. First of all, since the morphism $Y \to \operatorname{Spec} V$ is proper (especially, universally closed), the closure of any point of Y meets the closed fibre Y_0. It follows easily that the dimension of Y^{sing} is the maximum of the dimension of the local rings $\mathscr{O}_{Y^{\mathrm{sing}},y}$, where y ranges over all the points of $Y^{\mathrm{sing}} \cap Y_0$. However, $Y^{\mathrm{sing}} \cap Y_0$ is precisely the set of points $y \in Y_0$ with the property that Y_0 is not smooth over $\operatorname{Spec} V_0$ at y; in other words, $Y^{\mathrm{sing}} \cap Y_0 \subset \Sigma$, which consists of finitely many closed points, whence:

$$\dim \mathscr{O}_{Y^{\mathrm{sing}},y} \otimes_V V_0 = 0 \qquad \text{for every } y \in Y^{\mathrm{sing}} \cap Y_0.$$

On the other hand, quite generally we have the inequality:

$$\dim \mathscr{O}_{Y^{\mathrm{sing}},y} \leq \dim V + \dim \mathscr{O}_{Y^{\mathrm{sing}},y} \otimes_V V_0$$

for every $y \in Y^{\mathrm{sing}} \cap Y_0$ (see [41, Th.15.1]), therefore we conclude that the relative dimension of Y^{sing} over $\operatorname{Spec} V$ equals zero, hence Y^{sing} is finite over $\operatorname{Spec} V$, as claimed. Since U contains $Y^{\mathrm{sing}} \cap Y_0$ by construction, and since every point of Y^{sing} admits a specialization to a point of Y_0, it is clear that Y^{sing} is contained in U. ◊

For any V-algebra A denote by $A^\wedge_{\mathfrak{n}}$ the $\mathfrak{n}$-adic completion of A.

Claim 3.1.15. For any V-algebra R of finite type, the natural morphism $\operatorname{Spec} R^\wedge_{\mathfrak{n}} \to \operatorname{Spec} R$ is regular.

Proof of the claim. According to [20, Ch.IV, Prop.7.4.6], it suffices to show that all the formal fibres of R are geometrically regular. However, this follows from [20, Ch.IV, Th.7.4.4(ii)]. ◊

Say that $X = \operatorname{Spec} R$ and $U = \operatorname{Spec} S$, for some V-algebras R and S of finite type; (3.1.12) induces an isomorphism of V-schemes

$$\phi^\wedge : U^\wedge := \operatorname{Spec} S^\wedge_{\mathfrak{n}} \xrightarrow{\sim} X^\wedge := \operatorname{Spec} R^\wedge_{\mathfrak{n}}.$$

Claim 3.1.16. Let $x \in U^\wedge$ be any point whose image $y \in U$ lies outside Y^{sing}. Let $x' := \phi^\wedge(x)$ and denote by y' the image of x' in X. Then X is smooth over $\operatorname{Spec} V$ at the point y'.

Proof of the claim. It suffices to show that the induced morphism $\operatorname{Spec} \mathscr{O}_{X,y} \to \operatorname{Spec} V$ is regular ([22, Ch.IV, Cor.17.5.2]). However, by assumption $\operatorname{Spec} \mathscr{O}_{Y,y} \to \operatorname{Spec} V$ is regular, hence so is the morphism $\operatorname{Spec} \mathscr{O}_{Y^\wedge,x} \to \operatorname{Spec} V$, in view of claim 3.1.15. Consequently the morphism $\operatorname{Spec} \mathscr{O}_{X^\wedge,x'} \to \operatorname{Spec} V$ is regular, and since the map $\mathscr{O}_{X,y'} \to \mathscr{O}_{X^\wedge,x'}$ is faithfully flat, the claim follows from [41, Th.32.1]. ◊

Next, let $J \subset S$ be an ideal with $V(J) = Y^{\mathrm{sing}}$. In view of claim 3.1.14, the quotient S/J^n is finite over V for every $n \in \mathbb{N}$; especially, it is complete for the $\mathfrak{n}$-adic topology. We deduce from lemma 3.1.1(ii) that the natural map $S^\wedge_J \to S^\wedge_{\mathfrak{n}S+J}$ is an isomorphism (notation of (3.1)), and then lemma 3.1.1(i) implies that the natural map $S^\wedge_{\mathfrak{n}J} \to S^\wedge_{\mathfrak{n}}$ is an isomorphism as well, whence, by composing $\phi^\wedge$ with the natural map $X^\wedge \to X$, a morphism of V-schemes $\psi : \operatorname{Spec} S^\wedge_{\mathfrak{n}J} \to X$. The latter can be seen as a section

$$\sigma^\wedge : \operatorname{Spec} S^\wedge_{\mathfrak{n}J} \to U \times_{\operatorname{Spec} V} X$$

of the U-scheme $U \times_{\mathrm{Spec}\, V} X$ (namely, $\sigma^\wedge$ is the unique section such that $\pi \circ \sigma^\wedge = \psi$, where $\pi : U \times_{\mathrm{Spec}\, V} X \to X$ is the natural projection). It follows from claim 3.1.16 that the section $\sigma^\wedge$ fulfills the assumptions of [23, Ch.II, Th.2 bis], so that there exists a section

$$\sigma : \mathrm{Spec}\, S^h_{\mathfrak{n}J} \to U \times_{\mathrm{Spec}\, V} X.$$

such that σ and $\sigma^\wedge$ agree on the closed subscheme $\mathrm{Spec}\, S/\mathfrak{n}J$. Let $S^h_{\mathfrak{n}}$ be the henselization of S along the ideal $\mathfrak{n}S$; finally we define a morphism of V-schemes

$$\beta : \mathrm{Spec}\, S^h_{\mathfrak{n}} \xrightarrow{\omega} \mathrm{Spec}\, S^h_{\mathfrak{n}J} \xrightarrow{\sigma} U \times_{\mathrm{Spec}\, V} X \xrightarrow{\pi} X$$

where ω is the natural morphism. The theorem is now a straightforward consequence of the following :

Claim 3.1.17. β induces an isomorphism $\mathrm{Spec}\, S^h_{\mathfrak{n}} \xrightarrow{\omega} X^h_{/X_0}$.

Proof of the claim. By construction, β and $\phi^\wedge$ agree on the closed subscheme $\mathrm{Spec}\, S/\mathfrak{n}S$. Let $\beta^\wedge : U^\wedge \to X^\wedge$ be the morphism induced by β and denote by $\mathrm{gr}^\bullet_{\mathfrak{n}} R$ and $\mathrm{gr}^\bullet_{\mathfrak{n}} S$ the graded rings associated to the $\mathfrak{n}$-preadic filtrations of R and respectively S; we deduce that the morphisms β and $\phi^\wedge$ induce the same homomorphism $\mathrm{gr}^\bullet_{\mathfrak{n}} R \to \mathrm{gr}^\bullet_{\mathfrak{n}} S$ of graded rings. Since $\phi^\wedge$ is an isomorphism, it then follows that $\beta^\wedge$ is an isomorphism as well ([9, Ch.III, §2, n.8, Cor.3]). Next, according to [43, Ch.XI, §2, Th.2], the ring $S^h_{\mathfrak{n}}$ is the filtered colimit of a family of étale S-algebras $(S_\lambda \mid \lambda \in \Lambda)$ such that $S_\lambda/\mathfrak{n}S_\lambda$ is S-isomorphic to $S/\mathfrak{n}S$ for every $\lambda \in \Lambda$. Especially, the $\mathfrak{n}$-adic completion $S^\wedge_{\lambda,\mathfrak{n}}$ is S-isomorphic to $S^\wedge_{\mathfrak{n}}$, and we can find $\lambda \in \Lambda$ such that β descends to a morphism $\beta_\lambda : \mathrm{Spec}\, S_\lambda \to X$. Consequently, the map $R \to S^\wedge_{\lambda,\mathfrak{n}}$ is formally étale for the $\mathfrak{n}$-adic topology, therefore the same holds for the map $R \to S_\lambda$ induced by β_λ. Finally, let $\mathfrak{p} \in \mathrm{Spec}\, S_\lambda/\mathfrak{n}S_\lambda$, and set $\mathfrak{q} := \mathfrak{p} \cap R \in X_0$; *a fortiori* we see that $S_{\lambda,\mathfrak{q}}$ is formally étale over $R_{\mathfrak{q}}$ for the $\mathfrak{q}$-preadic and $\mathfrak{p}$-preadic topologies; by [22, Ch.IV, Prop.17.5.3] we deduce that S_λ is étale over R at all the points of $\mathrm{Spec}\, S_\lambda/\mathfrak{n}S_\lambda$. Since $S^h_{\mathfrak{n}} \simeq S^h_{\lambda,\mathfrak{n}}$, the claim follows. $\square$

3.1.18. Suppose that V is a valuation ring whose field of fractions K is algebraically closed, and let $S := \mathrm{Spec}\, V$. We conclude this section with a result stating the existence of semi-stable models for curves over the generic point of S. The proof consists in reducing to the case where V is noetherian and excellent, to which one can apply de Jong's method of alterations [12]. Recall that a *semi-stable S-curve* is a flat and proper morphism $g : Y \to S$ such that all the geometric fibres of g are connected curves having at most ordinary double points as singularities. Denote by η the generic point of S. We consider a projective finitely presented morphism $f : X \to S$ such that $f^{-1}(\eta)$ is irreducible of dimension one. We also assume that X is an integral scheme, and G is a given finite group of S-automorphisms of X.

Proposition 3.1.19. *In the situation of* (3.1.18), *there exists a projective and birational morphism $\phi : X' \to X$ of S-schemes such that :*

(a) *The structure morphism $f' : X' \to S$ is a semi-stable S-curve whose generic fibre $f'^{-1}(\eta)$ is irreducible and smooth over* $\mathrm{Spec}\, K$.
(b) *G acts on X' as a group of S-automorphisms, and ϕ is G-equivariant.*

Proof. Let us write V as the colimit of the filtered family $(V_i \mid i \in I)$ of its excellent noetherian subrings. By [21, Ch.IV, Th.8.8.2(i),(ii)], we may find $i \in I$ such that both f and the action of G descend to, respectively, a finitely presented morphism $f_i : X_i \to S_i := \mathrm{Spec}\, V_i$ and a finite group of S_i-automorphisms of X_i. By [21, Ch.IV, Th.8.10.5] we may even suppose that f_i is projective, and – up to replacing I by a cofinal subset – we may suppose that the latter property holds for all $i \in I$. For every $i \in I$, let η_i be the generic point of S_i; we apply [21, Ch.IV, Prop.8.7.2 and Cor.8.7.3] to the projective system of schemes $(f_i^{-1}(\eta_i) \mid i \in I)$ to deduce that there exists $i \in I$ such that $f_i^{-1}(\eta_i)$ is geometrically irreducible over $\mathrm{Spec}\, \kappa(\eta_i)$. Let $Z_i \subset X_i$ be

the Zariski closure (with reduced structure) of $f_i^{-1}(\eta_i)$; then $Z_i \times_{S_i} S$ is a closed subscheme of X containing $f^{-1}(\eta)$, so it coincides with X, since the latter is an integral scheme. Moreover, since f_i is a closed morphism and $\eta_i \in f_i(Z_i)$, we have $f_i(Z_i) = S_i$. Furthermore, the G-action on X_i restricts to a finite group of S_i-automorphisms of Z_i. The restriction $Z_i \to S_i$ of f_i fulfills the conditions of [12, Th.2.4], hence we may find a commutative diagram :

$$\begin{tikzcd} Z'_i \arrow[r, "\phi_i"] \arrow[d, "f'_i"'] & Z_i \arrow[d, "f_i"] \\ S'_i \arrow[r, "\psi_i"] & S_i \end{tikzcd}$$

such that Z'_i and S'_i are integral and excellent, ϕ_i and ψ_i are projective, dominant and generically finite, f'_i is a semi-stable projective S'_i-curve and moreover a finite group G' acts by S'_i-automorphisms on X'_i in such a way that f'_i is G'-equivariant; also there is a surjection $G' \to G$ so that ϕ_i is equivariant for the induced G'-action (condition (v) of [12, Th.2.4]). Furthermore, if η'_i denotes the generic point of S'_i, the induced morphism

$$f_i'^{-1}(\eta'_i) \to f_i^{-1}(\eta_i) \times_{\mathrm{Spec}\,\kappa(\eta_i)} \mathrm{Spec}\,\kappa(\eta'_i) \tag{3.1.20}$$

is birational (condition of [12, Th.2.4(vii)(b) and Rem.2.3(v)]). After taking the base change $S \to S_i$ we arrive at the commutative diagram :

$$\begin{tikzcd} Z' := Z'_i \times_{S_i} S \arrow[r, "\phi_{i,S}"] \arrow[d, "f'_{i,S}"'] & X \arrow[d, "f"] \\ S' := S'_i \times_{S_i} S \arrow[r, "\psi_{i,S}"] & S \end{tikzcd}$$

where again $\psi_{i,S}$ is generically finite and projective. Since $\kappa(\eta)$ is algebraically closed, it follows that the induced map $\kappa(\eta) \to \kappa(\eta')$ is bijective for every $\eta' \in \psi_{i,S}^{-1}(\eta)$; by the valuative criterion for properness ([17, Ch.II, Th.7.3.8(b)]) we deduce that $\psi_{i,S}$ admits a section $\sigma : S \to S'$. We set $X' := Z' \times_{S'} S$ (the base change along the morphism σ) and denote by $\phi : X' \to X$ the restriction of $\phi_{i,S}$. By construction, $f' : X' \to S$ is a semi-stable S-curve. From (3.1.20) we deduce easily that the induced morphism $f'^{-1}(\eta) \to f^{-1}(\eta)$ is birational, and then, since f' is flat, it follows that X' is integral and ϕ is birational. Moreover, the action of G' is completely determined by its restriction to the generic fibre $f'^{-1}(\eta)$, and since ϕ is equivariant, it follows that this action factors through G. □

3.2. **Vanishing cycles.** We resume that notation of (2.2); we let $S := \mathrm{Spec}\,K^+$, and denote by s the closed point of S. According to [30, §4.2], to every S-scheme X and every abelian sheaf F on $\mathsf{X}_{\text{ét}}$ one attaches a complex of abelian sheaves

$$R\Psi_{\mathsf{X}/S}(F) \in \mathsf{D}(\mathsf{X}_{s,\text{ét}}, \mathbb{Z})$$

where $\mathsf{X}_s := \mathsf{X} \times_S \mathrm{Spec}\,K^\sim$ and $\mathsf{D}(\mathsf{X}_{s,\text{ét}}, \mathbb{Z})$ denotes the derived category of the category of abelian sheaves on $\mathsf{X}_{s,\text{ét}}$. Its stalk at a point $x \in \mathsf{X}_s$ can be computed as follows. Denote by $\mathsf{X}^h_{/\{x\}}$ the spectrum of the henselization of the local ring $\mathscr{O}_{\mathsf{X},x}$; then there is a natural isomorphism in the derived category of complexes of abelian groups:

$$R\Psi_{\mathsf{X}/S}(F)_x \simeq R\Gamma((\mathsf{X}^h_{/\{x\}} \times_S \mathrm{Spec}\,K)_{\text{ét}}, F). \tag{3.2.1}$$

We let also $\mathsf{X}_K := \mathsf{X} \times_S \mathrm{Spec}\,K$. There is a natural map

$$R\Gamma(\mathsf{X}_K, F) \to R\Gamma(\mathsf{X}_s, R\Psi_{\mathsf{X}/S}(F))$$

which is an isomorphism when X is proper over S. On the other hand, one has a natural morphism in $\mathsf{D}(\mathsf{X}_{s,\text{ét}},\mathbb{Z})$

$$F_{|\mathsf{X}_s}[0] \to R\Psi_{\mathsf{X}/S}(F) \tag{3.2.2}$$

and the cone of (3.2.2) is a complex on X_s called the *complex of vanishing cycles* of the sheaf F, and denoted by $R\Phi_{\mathsf{X}/S}(F)$.

Any S-morphism $\phi : \mathsf{X} \to \mathsf{Y}$ induces a natural map of complexes

$$\phi^* R\Phi_{\mathsf{Y}/S}(F) \to R\Phi_{\mathsf{X}/S}(\phi^* F) \tag{3.2.3}$$

for every abelian sheaf F on $\mathsf{Y}_{\text{ét}}$.

3.2.4. Let $\pi \in \mathfrak{m}$ be any non-zero element, and set $\mathsf{A} := K^+[S,T]/(ST-\pi^2)$; the π-adic completion of A is the subring A° of the affinoid ring $A := A(a,a^{-1})$, where $a := |\pi|$ (cp. example 2.1.12). Let A^h be the henselization of A along its ideal $\pi\mathsf{A}$; by proposition 1.3.2(i), the base change functor :

$$\mathsf{A}^h\text{-}\mathbf{Alg}_{\text{fpét}/K} \to A^\circ\text{-}\mathbf{Alg}_{\text{fpét}/K} \quad : \quad \mathsf{B} \mapsto A^\circ \otimes_{\mathsf{A}^h} \mathsf{B}.$$

is an equivalence.

3.2.5. Let $f : X := \operatorname{Spa} B \to \mathbb{D}(a,a^{-1})$ be a finite étale morphism, so that B is a finite étale A-algebra, and suppose moreover that a finite group G acts freely on X in such a way that f becomes a G-equivariant morphism, provided we endow $\mathbb{D}(a,a^{-1})$ with the trivial G-action. This situation includes the basic case where f is a Galois (étale) morphism with Galois group G, but we also allow the case where G is the trivial group. Under these assumptions, B is normal, therefore the same holds for B°. Moreover, by lemma 2.3.1, B° is a finitely presented A°-module; we denote by B the unique (up to unique isomorphism) A^h-algebra corresponding to B° under the equivalence of (3.2.4). By proposition 1.3.2(iii), B is normal, and clearly the G-action on B translates into a G-action on B, fixing A^h. Since $\mathsf{A}^h/\mathfrak{m}\mathsf{A}^h \simeq A^\circ/\mathfrak{m}A^\circ$, we can view the prime ideal $\mathfrak{P} \subset A^\circ$ defined in the proof of claim 2.3.37, as an element of $\operatorname{Spec} \mathsf{A}^h/\mathfrak{m}\mathsf{A}^h$. Similarly, the finitely many prime ideals $\mathfrak{q}_1,\dots,\mathfrak{q}_n \subset B^\circ$ lying over $\mathfrak{P}$ can be viewed as elements of $\operatorname{Spec} \mathsf{B}/\mathfrak{m}\mathsf{B}$. We also obtain a natural action of G on the set $\{\mathfrak{q}_1,\dots,\mathfrak{q}_n\}$. For every $i = 1,\dots,n$, we let:

- $St(\mathfrak{q}_i) \subset G$ be the stabilizer of $\mathfrak{q}_i$,
- $\mathsf{A}^h_{\mathfrak{P}}$ (resp. B^h_i) the henselization of the local ring $\mathsf{A}_{\mathfrak{P}}$ (resp. of $\mathsf{B}_{\mathfrak{q}_i}$),
- $\mathsf{A}^h_{\mathfrak{P},K} := \mathsf{A}^h_{\mathfrak{P}} \otimes_{K^+} K$, $\mathsf{B}^h_{i,K} := \mathsf{B}^h_i \otimes_{K^+} K$, $\mathsf{T}^h_K := \operatorname{Spec} \mathsf{A}^h_{\mathfrak{P},K}$, $\mathsf{X}^h_{i,K} := \operatorname{Spec} \mathsf{B}^h_{i,K}$
- Λ a finite local ring such that $\ell^n\Lambda = 0$, for a prime number $\ell \neq \operatorname{char} K^\sim$ and some integer $n > 0$.

With this notation we define:

$$\Delta(X,\mathfrak{q}_i,F) := R\Gamma((\mathsf{X}^h_{i,K})_{\text{ét}},F) \quad \text{and} \quad \Delta(X,F) := \bigoplus_{i=1}^n \Delta(X,\mathfrak{q}_i,F)$$

for every sheaf of Λ-modules F on the étale site of $\mathsf{X}^h_{i,K}$. Moreover, if $C^\bullet$ is a bounded complex of Λ-modules and $H^\bullet C^\bullet$ is a finite Λ-module, we shall denote by $\chi(C^\bullet)$ the *Euler-Poincaré characteristic* of $C^\bullet$, which is defined by the rule :

$$\chi(C^\bullet) := \frac{\sum_{i\in\mathbb{Z}}(-1)^i \cdot \operatorname{length}_\Lambda(H^iC^\bullet)}{\operatorname{length}_\Lambda(\Lambda)}.$$

In case $C^\bullet$ is a complex of free Λ-modules (especially, $C^\bullet$ is perfect), we have also the identity:

$$\chi(C^\bullet) = \sum_{i\in\mathbb{Z}}(-1)^i \cdot \operatorname{rk}_\Lambda(C^i).$$

As usual ([32, Exp.X, §1]) one can view the constant sheaf $\Lambda_{\mathsf{X}^h_{i,K}}$ on $(\mathsf{X}^h_{i,K})_{\mathrm{\acute{e}t}}$ as a sheaf of G-modules with trivial G-action, and then by functoriality, $\Delta(X, \mathfrak{q}_i, \Lambda)$ is a complex of $\Lambda[St(\mathfrak{q}_i)]$-modules in a natural way. Furthermore, $\Delta(X, \Lambda)$ is a complex of $\Lambda[G]$-modules, whose structure can be analyzed as follows. Let $O_1, \cup \cdots \cup O_k = \{\mathfrak{q}_1, \dots, \mathfrak{q}_n\}$ be the decomposition into orbits under the G-action, and for every $i \leq k$ let us pick a representative $\mathfrak{q}_i \in O_i$; then :

$$\Delta(X, \Lambda) \simeq \sum_{i=1}^{k} \mathrm{Ind}^G_{St(\mathfrak{q}_i)} \Delta(X, \mathfrak{q}_i, \Lambda) \qquad \text{in } \mathsf{D}(\Lambda[G]\text{-}\mathbf{Mod}). \tag{3.2.6}$$

The proof is left to the reader.

Lemma 3.2.7. *With the notation of* (3.2.5)*, the following holds:*

(i) *The complex of* Λ*-modules* $\Delta(X, \mathfrak{q}_i, F)$ *is perfect of amplitude* $[0, 1]$*, for every constructible sheaf* F *of free* Λ*-modules on* $(\mathsf{X}^h_{i,K})_{\mathrm{\acute{e}t}}$.

(ii) *For every* $n \in \mathbb{N}$*, there is a natural isomorphism in* $\mathsf{D}^+(\mathbb{Z}/\ell^n\mathbb{Z}\text{-}\mathbf{Mod})$ *:*

$$\mathbb{Z}/\ell^n\mathbb{Z} \overset{\mathbf{L}}{\otimes}_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \Delta(X, \mathfrak{q}_i, \mathbb{Z}/\ell^{n+1}\mathbb{Z}) \xrightarrow{\sim} \Delta(X, \mathfrak{q}_i, \mathbb{Z}/\ell^n\mathbb{Z}).$$

(iii) *For every locally constant sheaf* F *of* Λ*-modules on* $(\mathsf{T}^h_K)_{\mathrm{\acute{e}t}}$ *we have :*

$$\chi(\Delta(\mathbb{D}(a, a^{-1}), \mathfrak{P}, F)) \leq 0.$$

Proof. To start out, notice that the scheme $\mathrm{Spec}\, \mathsf{B}^h_{\mathfrak{q},K}$ is a cofiltered limit of one-dimensional affine K-schemes of finite type; then (ii) is an easy consequence of [13, Th. finitude, Cor.1.11]. We also deduce that the cohomological dimension of $\mathrm{Spec}\, \mathsf{B}^h_{i,K}$ is ≤ 1; furthermore, it follows from (3.2.1) and [30, Prop.4.2.5] that $H^n\Delta(X, \mathfrak{q}_i, F)$ has finite length for every $n \in \mathbb{N}$ and every constructible sheaf F of Λ-modules, hence assertion (i) follows from :

Claim 3.2.8. Let R be a (not necessarily commutative) right noetherian ring with center $R_0 \subset R$, $\phi : \mathsf{Z} \to \mathsf{Y}$ a morphism of schemes, and $C^\bullet$ a complex in $\mathsf{D}^-(\mathsf{Z}_{\mathrm{\acute{e}t}}, R)$. Suppose that the functor

$$R\phi_* : \mathsf{D}^+(\mathsf{Z}_{\mathrm{\acute{e}t}}, R_0) \to \mathsf{D}^+(\mathsf{Y}_{\mathrm{\acute{e}t}}, R_0)$$

has finite cohomological dimension (thus, $R\phi_*$ extends to the whole of $\mathsf{D}(\mathsf{Z}_{\mathrm{\acute{e}t}}, R_0)$). Then :

(i) $C^\bullet$ is pseudo-coherent if and only if $H^nC^\bullet$ is coherent for every $n \in \mathbb{Z}$.

(ii) $C^\bullet$ is perfect if and only if it is pseudo-coherent and has locally finite Tor-dimension.

(iii) If the Tor-dimension of $C^\bullet$ is $\leq d$ (for some $d \in \mathbb{Z}$), then the Tor-dimension of $R\phi_*C^\bullet \in \mathrm{Ob}(\mathsf{D}^-(\mathsf{Y}_{\mathrm{\acute{e}t}}, R))$ is $\leq d$.

Proof of the claim. (i) (resp. (ii), resp. (iii)) is a special case of [5, Exp.I, Cor.3.5] (resp. [5, Exp.I, Cor.5.8.1], resp. [4, Exp.XVII, Th.5.2.11]). ◊

(iii): It follows from (i) that the Euler-Poincaré characteristic of $\Delta(\mathbb{D}(a, a^{-1}), \mathfrak{P}, F)$ is well defined when Λ is a finite field of characteristic ℓ. For the general case, let $\mathfrak{m}_\Lambda \subset \Lambda$ be the maximal ideal; we consider the descending filtration $F \supset \mathfrak{m}_\Lambda F \supset \mathfrak{m}^2_\Lambda F \supset \cdots \supset \mathfrak{m}^r_\Lambda F = 0$, whose graded subquotients are sheaves of modules over the residue field $\kappa(\Lambda)$ of Λ; since the expression that we have to evaluate is obviously additive in F, we are then reduced to the case where F is an irreducible locally constant sheaf of $\kappa(\Lambda)$-modules. In such case, if F is not constant the assertion is clear, so we can further suppose that F is the constant sheaf with stalks isomorphic to $\kappa(\Lambda)$. However, $\mathsf{A}^h_{\mathfrak{P}}$ is a normal domain, therefore T^h_K is connected, so $H^0\Delta(\mathbb{D}(a, a^{-1}), \mathfrak{P}, \kappa(\Lambda)) = \kappa(\Lambda)$. Finally, from [14, Exp.XV, §2.2.5] we derive $H^1\Delta(\mathbb{D}(a, a^{-1}), \mathfrak{P}, \kappa(\Lambda)) = \kappa(\Lambda)$. (*loc.cit.* considers the vanishing cycle functor relative to a family defined over a strictly henselian discrete valuation ring, but by inspecting the proofs it is easy to see that the same argument works *verbatim* in our setting as well.) □

3.2.9. Let R be a (not necessarily commutative) ring. We denote by $K^0(R)$ (resp. by $K_0(R)$) the Grothendieck group of finitely generated projective (resp. finitely presented) left R-modules. Any perfect complex $C^\bullet$ of R-modules determines a class $[C^\bullet] \in K^0(R)$. Namely, one chooses a quasi-isomorphism $P^\bullet \xrightarrow{\sim} C^\bullet$ from a bounded complex of finitely generated projective left R-modules, and sets $[C^\bullet] := \sum_{i\in\mathbb{Z}}(-1)^i \cdot [P^i]$; a standard verification shows that the definition does not depend on the chosen projective resolution.

Proposition 3.2.10. *In the situation of* (3.2.5) *we have:*

(i) $\Delta(X, \mathfrak{q}_i, \Lambda)$ *is a perfect complex of* $\Lambda[St(\mathfrak{q}_i)]$*-modules of amplitude* $[0, 1]$.
(ii) $[\Delta(X, \mathfrak{q}_i, \mathbb{F}_\ell)[1]] \in K^0(\mathbb{F}_\ell[St(\mathfrak{q}_i)])$ *is the class of a projective* $\mathbb{F}_\ell[St(\mathfrak{q}_i)]$*-module.*
(iii) $[\Delta(X, \mathbb{F}_\ell)[1]] \in K^0(\mathbb{F}_\ell[G])$ *is the class of a projective* $\mathbb{F}_\ell[G]$*-module.*

Proof. (i): This is well known: let $\mathsf{f}_K : \mathsf{X}^h_{i,K} \to \mathsf{T}^h_K$ be the natural morphism; one shows as in the proof of [32, Exp.X, Prop.2.2] that $\mathsf{f}_{K*}\Lambda$ is a flat sheaf of $\Lambda[St(\mathfrak{q}_i)]$-modules, and then claim 3.2.8(iii) implies that the complex of $\Lambda[St(\mathfrak{q}_i)]$-modules $\Delta(X, \mathfrak{q}_i, \Lambda) \simeq \Delta(\mathbb{D}(a, a^{-1}), \mathfrak{P}, \mathsf{f}_{K*}\Lambda)$ is of finite Tor-dimension. It then follows from claim 3.2.8(i),(ii) and lemma 3.2.7(i) that $\Delta(X, \mathfrak{q}_i, \Lambda)$ is a perfect complex of $\Lambda[St(\mathfrak{q}_i)]$-modules of amplitude $[0, 1]$.

(ii): Let us choose a complex $\Delta^\bullet := (\Delta^0 \to \Delta^1)$ of finitely generated projective $\mathbb{F}_\ell[St(\mathfrak{q}_i)]$-modules with a quasi-isomorphism $\Delta^\bullet \xrightarrow{\sim} \Delta(X, \mathfrak{q}_i, \mathbb{F}_\ell)$; if M is any $\mathbb{F}_\ell[St(\mathfrak{q}_i)]$-module of finite length, we deduce a quasi-isomorphism (cp. the proof of lemma 3.2.7(ii))

$$M \otimes_{\mathbb{F}_\ell[St(\mathfrak{q}_i)]} \Delta^\bullet \xrightarrow{\sim} R\Gamma(\mathsf{T}^h_K, M \otimes_{\mathbb{F}_\ell[St(\mathfrak{q}_i)]} \mathsf{f}_{K*}\mathbb{F}_\ell).$$

Whence $\chi(M \otimes_{\mathbb{F}_\ell[St(\mathfrak{q}_i)]} \Delta^\bullet) \leq 0$, in view of lemma 3.2.7(iii). In other words :

$$\mathrm{rk}_{\mathbb{F}_\ell} M \otimes_{\mathbb{F}_\ell[St(\mathfrak{q}_i)]} \Delta^0 \leq \mathrm{rk}_{\mathbb{F}_\ell} M \otimes_{\mathbb{F}_\ell[St(\mathfrak{q}_i)]} \Delta^1 \qquad \text{for every } M \text{ of finite length.}$$

On the other hand, $K_0(\mathbb{F}_\ell[St(\mathfrak{q}_i)])$ is endowed with an involution ([32, Exp.X, §3.7])

$$K_0(\mathbb{F}_\ell[St(\mathfrak{q}_i)]) \to K_0(\mathbb{F}_\ell[St(\mathfrak{q}_i)]) \quad : \quad [M] \mapsto [M]^* := [M^*] := [\mathrm{Hom}_{\mathbb{F}_\ell}(M, \mathbb{F}_\ell)].$$

We have a natural isomorphism

$$\mathrm{Hom}_{\mathbb{F}_\ell[St(\mathfrak{q}_i)]}(N, M^*) \simeq (N \otimes_{\mathbb{F}_\ell[St(\mathfrak{q}_i)]} M)^*$$

for every $\mathbb{F}_\ell[St(\mathfrak{q}_i)]$-modules of finite length M and N ([32, Exp.X, Prop.3.8]). Since clearly $\mathrm{rk}_{\mathbb{F}_\ell} M = \mathrm{rk}_{\mathbb{F}_\ell} M^*$ for every such M, we conclude that

(3.2.11)
$$\mathrm{rk}_{\mathbb{F}_\ell}\mathrm{Hom}_{\mathbb{F}_\ell[St(\mathfrak{q}_i)]}(\Delta^0, M) \leq \mathrm{rk}_{\mathbb{F}_\ell}\mathrm{Hom}_{\mathbb{F}_\ell[St(\mathfrak{q}_i)]}(\Delta^1, M) \quad \text{for every } M \text{ of finite length.}$$

By [47, §14.3, Cor.1,2] the projective modules Δ^0 and Δ^1 are direct sums of projective envelopes of simple $\mathbb{F}_\ell[St(\mathfrak{q}_i)]$-modules; however, (3.2.11) implies that the multiplicity in Δ^0 of the projective envelope P_N of any simple module N is $\leq$ the multiplicity of P_N in Δ^1, whence the assertion.

(iii) is an easy consequence of (ii) and (3.2.6). $\square$

3.2.12. Keep the assumptions of proposition 3.2.10. For every $n \in \mathbb{N}$ we set $\Lambda_n := \mathbb{Z}/\ell^n\mathbb{Z}$; in view of lemma 3.2.7(ii) we derive natural isomorphisms in $\mathsf{D}^+(\Lambda_n[St(\mathfrak{q}_i)]\text{-}\mathbf{Mod})$:

$$\Lambda_n[St(\mathfrak{q}_i)] \overset{\mathbf{L}}{\otimes}_{\Lambda_{n+1}[St(\mathfrak{q}_i)]} \Delta(X, \mathfrak{q}_i, \Lambda_{n+1}) \xrightarrow{\sim} \Lambda_n \overset{\mathbf{L}}{\otimes}_{\Lambda_{n+1}} \Delta(X, \mathfrak{q}_i, \Lambda_{n+1}) \xrightarrow{\sim} \Delta(X, \mathfrak{q}_i, \Lambda_n). \tag{3.2.13}$$

Then, according to [32, Exp.XIV, §3, n.3, Lemme 1] we may find :

- An inverse system $(\Delta^\bullet_n(X, \mathfrak{q}_i) \mid n \in \mathbb{N})$, such that $\Delta^\bullet_n(X, \mathfrak{q}_i)$ is a complex of projective $\Lambda_n[St(\mathfrak{q}_i)]$-modules of finite rank, concentrated in degrees 0 and 1, and the transition maps are isomorphisms of complexes of $\Lambda_n[St(\mathfrak{q}_i)]$-modules :

$$\Lambda_n \otimes_{\Lambda_{n+1}} \Delta^\bullet_{n+1}(X, \mathfrak{q}_i) \xrightarrow{\sim} \Delta^\bullet_n(X, \mathfrak{q}_i) \qquad \text{for every } n \in \mathbb{N}. \tag{3.2.14}$$

- A system of isomorphisms : $\Delta^\bullet_n(X,\mathfrak{q}_i) \xrightarrow{\sim} \Delta(X,\mathfrak{q}_i,\Lambda_n)$ in $\mathsf{D}^+(\Lambda_n[St(\mathfrak{q}_i)]\text{-}\mathbf{Mod})$, compatible with the isomorphisms (3.2.13) and (3.2.14).

(Actually, *loc.cit.* includes the assumption that the coefficient rings are commutative. This assumption is not verified by our system of rings $\Lambda_n[St(\mathfrak{q}_i)]$; however, by inspecting the proof, one sees easily that the commutativity is not needed.)

We let $\Delta^\bullet_\infty(X,\mathfrak{q}_i)$ be the inverse limit of the system $(\Delta^\bullet_n(X,\mathfrak{q}) \mid n\in\mathbb{N})$; this is a complex of projective $\mathbb{Z}_\ell[St(\mathfrak{q}_i)]$-modules of finite rank, concentrated in degrees 0 and 1, and we have isomorphisms of complexes of $\Lambda_n[St(\mathfrak{q}_i)]$-modules :

$$\Lambda_n[St(\mathfrak{q}_i)]\otimes_{\mathbb{Z}_\ell[St(\mathfrak{q}_i)]}\Delta^\bullet_\infty(X,\mathfrak{q}_i)\simeq\Lambda_n\otimes_{\mathbb{Z}_\ell}\Delta^\bullet_\infty(X,\mathfrak{q}_i)\simeq\Delta^\bullet_n(X,\mathfrak{q}_i)\qquad\text{for every } n\in\mathbb{N}.$$

Likewise, we set

$$\Delta^\bullet_n(X):=\bigoplus_{i=1}^n\Delta^\bullet_n(X,\mathfrak{q}_i)$$

and the analogue of (3.2.6) holds for $\Delta^\bullet_n(X)$, especially the latter is a complex of finitely generated projective $\Lambda_n[G]$-modules and the inverse limit $\Delta^\bullet_\infty(X)$ of the system $(\Delta^\bullet_n(X)\mid n\in\mathbb{N})$ is a complex of finitely generated projective $\mathbb{Z}_\ell[G]$-modules.

Lemma 3.2.15. *In the situation of* (3.2.12) *:*

(i) *The element* $[\Delta^\bullet_\infty(X,\mathfrak{q}_i)[1]]\in K^0(\mathbb{Z}_\ell[St(\mathfrak{q}_i)]\text{-}\mathbf{Mod})$ *is the class of a finitely generated projective* $\mathbb{Z}_\ell[St(\mathfrak{q}_i)]$*-module.*

(ii) *The element* $[\Delta^\bullet_\infty(X)[1]]\in K^0(\mathbb{Z}_\ell[G]\text{-}\mathbf{Mod})$ *is the class of a finitely generated projective* $\mathbb{Z}_\ell[G]$*-module.*

Proof. (i) follows from proposition 3.2.10(ii) and [47, §14.4, Cor.3], and (ii) follows from (i). □

3.2.16. In view of lemma 3.2.15, the element $[\Delta^\bullet_\infty(X,\mathfrak{q}_i)\otimes_\mathbb{Z}\mathbb{Q}]\in K^0(\mathbb{Q}_\ell[St(\mathfrak{q}_i)])$ is the class of a finite-dimensional ℓ-adic representation of $St(\mathfrak{q}_i)$. For such representations ρ, it makes sense to ask whether the associated character takes only rational values, *i.e.* whether the class $[\rho]$ lies in the subgroup $\overline{R}_\mathbb{Q}(St(\mathfrak{q}_i))\subset K^0(\mathbb{Q}_\ell[St(\mathfrak{q}_i)])$ (notation of [47, §12.1]), and a complete charaterization of $\mathbb{Q}\otimes_\mathbb{Z}\overline{R}_\mathbb{Q}(St(\mathfrak{q}_i))$ is provided by the criterion of [47, §13.1]. The availability of that criterion is the main reason why we are interested in ℓ-adic representations (rather than just ℓ-torsion ones).

Theorem 3.2.17. *With the notation of* (3.2.16) *:*

(i) *The class* $[\Delta^\bullet_\infty(X,\mathfrak{q}_i)\otimes_\mathbb{Z}\mathbb{Q}]$ *lies in* $\overline{R}_\mathbb{Q}(St(\mathfrak{q}_i))$.

(ii) *The class* $[\Delta^\bullet_\infty(X)\otimes_\mathbb{Z}\mathbb{Q}]$ *lies in* $\overline{R}_\mathbb{Q}(G)$.

Proof. Of course it suffices to show (i). We begin with the following :

Claim 3.2.18. There exist :

- a projective birational morphism $\phi:\mathsf{Y}\to\mathsf{X}$ of integral projective S-schemes, where $\mathsf{Y}\to S$ is a semistable S-curve with smooth connected generic fibre $\mathsf{Y}_K\to\operatorname{Spec}K$;
- group homomorphisms $St(\mathfrak{q}_i)\to\mathrm{Aut}_S\mathsf{Y}$ and $St(\mathfrak{q}_i)\to\mathrm{Aut}_S\mathsf{X}$ such that ϕ is $St(\mathfrak{q}_i)$-equivariant;
- a point $x\in\mathsf{X}_s$, fixed by $St(\mathfrak{q}_i)$, with an $St(\mathfrak{q}_i)$-equivariant isomorphism : $\mathscr{O}^h_{\mathsf{X},x}\xrightarrow{\sim}\mathsf{B}^h_i$;
- an open neighborhood $\mathsf{U}\subset\mathsf{X}$ of x, which is a connected normal scheme.

Proof of the claim. To start out, B^h_i is the colimit of a filtered family $(\mathsf{B}_\mu\mid\mu\in J)$ of étale B-algebras, and since since B^h_i is a normal domain, we may assume that the same holds for every B_μ. We may find $\mu\in J$ such that the action of $St(\mathfrak{q}_i)$ on B^h_i descends to an action by K^+-automorphisms on B_μ ([21, Ch.IV, Th.8.8.2(i)]). If $x\in\mathsf{U}:=\operatorname{Spec}\mathsf{B}_\mu$ denotes the contraction

of the ideal $\mathfrak{q}_i \subset \mathsf{B}$, we have an $St(\mathfrak{q}_i)$-equivariant isomorphism $\mathscr{O}^h_{\mathsf{U},x} \xrightarrow{\sim} \mathsf{B}^h_i$. Let $V \subset \mathsf{B}_\mu$ be a finitely generated K^+-submodule, say of rank $n+1$, that generates $\mathscr{O}_\mathsf{U}$; V determines a morphism $\psi : \mathsf{U} \to \mathbb{P}^n_S$, and by choosing V large enough, we may achieve that ψ is a closed immersion; moreover we may suppose that V is stable under the natural $St(\mathfrak{q}_i)$-action, in which case ψ is $St(\mathfrak{q}_i)$-equivariant. Denote by X the Zariski closure of $\psi(\mathsf{U})$ (with reduced structure). Then the action of $St(\mathfrak{q}_i)$ extends to X and $\mathsf{f} : \mathsf{X} \to S$ is a projective finitely presented morphism; moreover, since U is a normal scheme, the generic fibre $\mathsf{f}^{-1}(\eta)$ is irreducible. To conclude, it suffices to invoke proposition 3.1.19. $\diamondsuit$

Hence, let $\phi : \mathsf{Y} \to \mathsf{X}$ and $x \in \mathsf{U} \subset \mathsf{X}_s$ be as in claim 3.2.18, and $\phi_s : \mathsf{Y}_s \to \mathsf{X}_s$ (resp. $\phi_\eta : \mathsf{Y}_K \to \mathsf{X}_K$) the restriction of ϕ; applying the proper base change theorem ([4, Exp.XII, Th.5.1]), we derive a natural $St(\mathfrak{q}_i)$-equivariant isomorphism :

$$R\phi_{s*}R\Psi_{\mathsf{Y}/S}\Lambda \xrightarrow{\sim} R\Psi_{\mathsf{X}/S}R\phi_{\eta*}\Lambda \tag{3.2.19}$$

in $\mathsf{D}(\mathsf{X}_{s,\text{ét}}, \Lambda)$, for every ring Λ as in (3.2.5). Set $Z := \phi_s^{-1}(x)$; taking the stalk over x of the map (3.2.19) yields an $St(\mathfrak{q}_i)$-equivariant isomorphism :

$$R\Gamma(Z, R\Psi_{\mathsf{Y}/S}\Lambda) \xrightarrow{\sim} (R\Psi_{\mathsf{X}/S}R\phi_{\eta*}\Lambda)_x. \tag{3.2.20}$$

Moreover, U_K is smooth over $\mathrm{Spec}\, K$, hence ϕ_η restricts to an isomorphism on $\phi^{-1}\mathsf{U}_K$, hence:

$$(R\Psi_{\mathsf{X}/S}R\phi_{\eta*}\Lambda)_x \simeq (R\Psi_{\mathsf{X}/S}\Lambda)_x \simeq \Delta(X, \mathfrak{q}_i, \Lambda) \tag{3.2.21}$$

where again these isomorphisms are $St(\mathfrak{q}_i)$-equivariant. On the other hand, [14, Exp.XV, §2.2] yields natural isomorphisms :

$$R^0\Psi_{\mathsf{Y}/S}\Lambda \simeq \Lambda_{|\mathsf{Y}_s} \qquad R^1\Psi_{\mathsf{Y}/S}\Lambda \simeq i_*\Lambda(-1)_{|\mathsf{Y}^{\mathrm{sing}}_s} \qquad R^j\Psi_{\mathsf{Y}/S}\Lambda = 0 \quad \text{for } j > 1 \tag{3.2.22}$$

where $i : \mathsf{Y}^{\mathrm{sing}}_s \to \mathsf{Y}_s$ is the closed immersion of the singular locus of Y_s (which consists of finitely many $\mathrm{Spec}\, K^\sim$-rational points) and (-1) denotes the Tate twist. (Actually, *loc.cit.* considers the case where S is a henselian discrete valuation ring, but by inspecting the proof one sees easily that the same argument works in our situation as well.) Since Z is proper over $\mathrm{Spec}\, K^\sim$, one may apply [4, Exp.XVII, Th.5.4.3] to deduce that (3.2.20) and (3.2.21) still hold after we replace Λ by $\mathbb{Q}_\ell$ and $\Delta(X, \mathfrak{q}_i, \Lambda)$ by $\Delta^\bullet_\infty(X, \mathfrak{q}_i) \otimes_\mathbb{Z} \mathbb{Q}$. Hence, $[\Delta^\bullet_\infty(X, \mathfrak{q}_i) \otimes_\mathbb{Z} \mathbb{Q}]$ is the difference of the classes :

$$R_1 := [R\Gamma(Z, \mathbb{Q}_\ell)] \qquad R_2 := [\Gamma(Z^{\mathrm{sing}}, \mathbb{Q}_\ell(-1))].$$

where the $St(\mathfrak{q}_i)$-actions on R_1 and R_2 are deduced by functoriality from the actions of $St(\mathfrak{q}_i)$ on the sheaves $\mathbb{Q}_\ell$ and $\mathbb{Q}_\ell(-1)$, and the latter are defined via (3.2.22). So the theorem follows from the following :

Claim 3.2.23. $R_1, R_2 \in \overline{R}_\mathbb{Q}(St(\mathfrak{q}_i))$.

Proof of the claim. Concerining R_1 : first of all, notice that the action of $St(\mathfrak{q}_i)$ on $\Lambda_{|\mathsf{Y}_s}$ (resp. on $\mathbb{Q}_{\ell|\mathsf{Y}_s}$) induced by the isomorphism (3.2.22), is the trivial one (this isomorphism is the map (3.2.2)). Let $\rho : Z' \to Z$ be the normalization morphism, and say that $W_1, \dots, W_k$ are the irreducible components of Z'; a standard *dévissage* shows that

$$R_1 = [R\Gamma_c(Z', \mathbb{Q}_\ell)] - [\Gamma(\rho^{-1}Z^{\mathrm{sing}}, \mathbb{Q}_\ell)] + [\Gamma(Z^{\mathrm{sing}}, \mathbb{Q}_\ell)]$$

where the $St(\mathfrak{q}_i)$-actions on the terms appearing on the right-hand side are deduced, by functoriality, from the trivial $St(\mathfrak{q}_i)$-actions on the constant ℓ-adic sheaves $\mathbb{Q}_\ell$ on the scheme Z'. Let $g \in St(\mathfrak{q}_i)$ be any element, $g' : Z' \to Z'$ the unique $K^\sim$-automorphism that lifts the action of g on Z, and $g'' : g'^*\mathbb{Q}_\ell \xrightarrow{\sim} \mathbb{Q}_\ell$ the isomorphism that defines the trivial $St(\mathfrak{q}_i)$-action on $\mathbb{Q}_{\ell|Z'}$. We

have a natural decomposition : $R\Gamma_c(Z', \mathbb{Q}_\ell) \simeq \bigoplus_{i=1}^k R\Gamma_c(W_i, \mathbb{Q}_\ell)$, and $R\Gamma_c(Z', g'')$ restricts to isomorphisms :

$$\omega_i : R\Gamma_c(g'(W_i), \mathbb{Q}_\ell) \xrightarrow{\sim} R\Gamma_c(W_i, \mathbb{Q}_\ell) \qquad \text{for every } i = 1, \dots, k.$$

It follows that the trace of $R\Gamma_c(Z', g'')$ is the sum of the traces of the maps ω_i such that $g'(W_i) = W_i$. Each W_i is either a point or a smooth projective $K^\sim$-curve. In case W_i is a point, ω_i is the identity map; to determine the trace of ω_i in case W_i is a smooth curve, we may apply the Lefschetz fixed point formula [13, Rapport, Th.5.3], and it follows easily that $[R\Gamma_c(Z', \mathbb{Q}_\ell)] \in \overline{R}_\mathbb{Q}(St(\mathfrak{q}_i))$. Next, consider the term $[\Gamma(Z^{\mathrm{sing}}, \mathbb{Q}_\ell)]$; a similar argument shows that, in order to compute the trace of the automorphism induced by g on $\Gamma(Z^{\mathrm{sing}}, \mathbb{Q}_\ell)$, we may neglect all the points of Z^{sing} that are not fixed by the action of g; if $z \in Z^{\mathrm{sing}}$ is fixed by g, then clearly the trace of $\Gamma(\{z\}, g)$ equals 1, so we get as well $[\Gamma(Z^{\mathrm{sing}}, \mathbb{Q}_\ell)] \in \overline{R}_\mathbb{Q}(St(\mathfrak{q}_i))$. Finally, we consider $[\Gamma(\rho^{-1}Z^{\mathrm{sing}}, \mathbb{Q}_\ell)]$: let $(Z^{\mathrm{sing}})^g$ be the set of points of Z^{sing} that are fixed by g; an argument as in the foregoing shows that the trace of $\Gamma(\rho^{-1}Z^{\mathrm{sing}}, g'')$ is the same as the trace of $\Gamma(\rho^{-1}(Z^{\mathrm{sing}})^g, g'')$. For every $z \in (Z^{\mathrm{sing}})^g$, the fibre $\rho^{-1}(z)$ consists of two points z'_1 and z'_2, and clearly g either exchanges them, in which case the corresponding contribution to the trace is 0, or else g fixes them, in which case the contribution is 2. Hence $[\Gamma(\rho^{-1}Z^{\mathrm{sing}}, \mathbb{Q}_\ell)] \in \overline{R}_\mathbb{Q}(St(\mathfrak{q}_i))$, so the claim holds for R_1.

Concerning R_2 : let again $g \in St(\mathfrak{q}_i)$ be any element; the action of g on R_2 is induced by an action of g on $(R^1\Psi_\eta\mathbb{Q}_\ell)_{|\mathsf{Y}^{\mathrm{sing}}}$, *i.e.* by an isomorphism $g' : g^*\mathbb{Q}_\ell(-1)_{|\mathsf{Y}^{\mathrm{sing}}} \xrightarrow{\sim} \mathbb{Q}_\ell(-1)_{|\mathsf{Y}^{\mathrm{sing}}}$. Arguing as in the foregoing case, we see that the trace of $\Gamma(Z^{\mathrm{sing}}, g')$ is the same as the trace of $\Gamma((Z^{\mathrm{sing}})^g, g')$. Hence, for our purposes, it suffices to determine the automorphism g'_z of the stalk over any $z \in Z^{\mathrm{sing}}$ which is fixed by g. By [14, Exp.XV, §2.2], Poincaré duality yields a perfect pairing :

$$(R^1\Psi_\eta\mathbb{Q}_\ell)_z \times H^1_{\{z\}}(\mathsf{Y}_s, R\Psi_\eta\mathbb{Q}_\ell(1)) \to \mathbb{Q}_\ell.$$

Hence it suffices to show that $[H^1_{\{z\}}(\mathsf{Y}_s, R\Psi_\eta\mathbb{Q}_\ell(1))]$ lies in $\overline{R}_\mathbb{Q}(St(\mathfrak{q}_i))$. However, according to [14, Exp.XV, lemme 2.2.7], we have a natural short exact sequence :

$$0 \to H^0(\{z\}, \mathbb{Q}_\ell)(1) \to H^0(\rho^{-1}\{z\}, \mathbb{Q}_\ell)(1) \to H^1_{\{z\}}(\mathsf{Y}_s, R\Psi_\eta\mathbb{Q}_\ell(1)) \to 0$$

which becomes $St(\mathfrak{q}_i)$-equivariant, provided we endow the ℓ-adic sheaves $\mathbb{Q}_{\ell|\{z\}}$ and $\mathbb{Q}_{\ell|\rho^{-1}\{z\}}$ with their trivial actions. It follows that the trace of g'_z equals 1 if g fixes the points of $\rho^{-1}\{z\}$, and equals -1 if g exchanges these two points. □

3.2.24. Keep the notation of (3.2.5) and let $H \subset G$ be any subgroup; since A is a Japanese ring ([6, §6.1.2, Prop.4]) we see easily that the subring B^H of elements fixed by H is an affinoid algebra; we can then consider the morphism $f_H : X/H := \mathrm{Spa}\, B^H \to \mathbb{D}(a, a^{-1})$; clearly f_H is again étale (indeed, this can be checked after an étale base change, especially, after base change to X, in which case the assertion is obvious). Moreover, obviously $(B^H)^\circ = (B^\circ)^H$; under the equivalence of (3.2.4), the finitely presented A°-algebra $(B^H)^\circ$ corresponds to a unique (up to unique isomorphism) finitely presented A^h-algebra C such that $\mathsf{C} \otimes_{K^+} K$ is étale over $\mathsf{A}^h_K := \mathsf{A}^h \otimes_{K^+} K$. By lemma 1.3.6 the natural map $\mathsf{A}^h \to A^\circ$ is faithfully flat; by considering the left exact sequence of A^h-modules

$$0 \longrightarrow \mathsf{B}^H \longrightarrow \mathsf{B} \xrightarrow{\oplus_{h\in H}(1-h)} \bigoplus_{h\in H} \mathsf{B}$$

one deduces easily that

$$\mathsf{C} = \mathsf{B}^H. \tag{3.2.25}$$

3.2.26. Suppose next, that the subgroup $H \subset G$ is contained in $St(\mathfrak{q}_i)$. We denote by $\mathfrak{q}_i^H$ the image of $\mathfrak{q}_i$ in $\mathrm{Spec}\,(B^H)^\circ$. In view of (3.2.25) the induced map

$$\mathsf{g} : \mathsf{X}^h_{i,K} \to \mathsf{Y}^h_{i,K} := \mathrm{Spec}\,\mathsf{C}^h_{\mathfrak{q}_i^H,K}$$

is a Galois étale covering with Galois group H.

Lemma 3.2.27. *In the situation of* (3.2.26)*, we have a natural isomorphism in* $\mathsf{D}(\mathbb{Z}_\ell\text{-}\mathbf{Mod})$ *:*

$$\Delta^\bullet_\infty(X/H, \mathfrak{q}_i^H) \xrightarrow{\sim} \Delta^\bullet_\infty(X, \mathfrak{q}_i)^H.$$

Proof. (Notice that, since in general H is not a normal subgroup of $St(\mathfrak{q}_i)$, the only group surely acting on $\Delta(X/H, \mathfrak{q}_i^H, \Lambda_n)$ is the trivial one, so $\Delta^\bullet_\infty(X/H, \mathfrak{q}_i^H)$ is to be meant as a complex of free $\mathbb{Z}_\ell$-modules.) To start with, let Λ be any ring as in (3.2.5); the functor $F \mapsto \underline{\Gamma}^H(F) := F^H$ on sheaves of $\Lambda[H]$-modules on $(\mathsf{Y}^h_{i,K})_{\text{ét}}$ induces a derived functor

$$R\underline{\Gamma}^H : \mathsf{D}^+((\mathsf{Y}^h_{i,K})_{\text{ét}}, \Lambda[H]) \to \mathsf{D}^+((\mathsf{Y}^h_{i,K})_{\text{ét}}, \Lambda).$$

Likewise, we have a derived functor :

$$R\Gamma^H : \mathsf{D}^+(\Lambda[H]\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\Lambda\text{-}\mathbf{Mod}).$$

Especially, consider the inverse system $(\Delta^\bullet_n(X, \mathfrak{q}_i) \mid n \in \mathbb{N})$ of (3.2.12); since each $\Delta^\bullet_n(X, \mathfrak{q}_i)$ is a complex of projective $\Lambda_n[St(\mathfrak{q}_i)]$-modules, the natural map :

$$\Gamma^H \Delta^\bullet_n(X, \mathfrak{q}_i) \to R\Gamma^H \Delta^\bullet_n(X, \mathfrak{q}_i)$$

is an isomorphism in $\mathsf{D}(\Lambda_n\text{-}\mathbf{Mod})$, for every $n \in \mathbb{N}$. Similarly, since $\mathsf{g}_*\Lambda_{n,\mathsf{X}^h_{i,K}}$ is a sheaf of projective $\Lambda_n[H]$-modules, we have natural isomorphisms of sheaves on $(\mathsf{Y}^h_{i,K})_{\text{ét}}$:

$$\Lambda_{n,\mathsf{Y}^h_{i,K}} \xrightarrow{\sim} R\underline{\Gamma}^H \mathsf{g}_*\Lambda_{n,\mathsf{X}^h_{i,K}}. \tag{3.2.28}$$

Now, by applying to (3.2.28) the triangulated functor

$$R\Gamma : \mathsf{D}((\mathsf{Y}^h_{i,K})_{\text{ét}}, \Lambda_n) \to \mathsf{D}(\Lambda_n\text{-}\mathbf{Mod})$$

and using the obvious isomorphism of triangulated functors

$$R\Gamma \circ R\underline{\Gamma}^H \simeq R\Gamma^H \circ R\Gamma : \mathsf{D}^+((\mathsf{Y}^h_{i,K})_{\text{ét}}, \Lambda_n[H]) \to \mathsf{D}^+(\Lambda_n\text{-}\mathbf{Mod})$$

we deduce natural isomorphisms :

$$\Delta^\bullet_n(X/H, \mathfrak{q}_i^H) \xrightarrow{\sim} R\Gamma((\mathsf{Y}^h_{i,K})_{\text{ét}}, \Lambda_n) \xrightarrow{\sim} \Gamma^H \Delta^\bullet_n(X, \mathfrak{q}_i) \qquad \text{for every } n \in \mathbb{N}.$$

The assertion then follows after taking inverse limits. $\square$

3.2.29. In the situation of (3.2.5), let $\delta : [\log a, -\log a] \cap \log \Gamma_K \to \mathbb{R}_{\geq 0}$ be the discriminant function of the morphism f. We saw in the course of the proof of theorem 2.3.35 how one can calculate the variation of the slope of δ at the point $\rho = 0$ – that is, the slope around $\rho = 0$ of the function $\rho \mapsto \delta(-\rho) + \delta(\rho)$. The expression is a sum of contributions indexed by the prime ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_n$. Proposition 3.2.30 explains how these localized contributions can be read off from the complexes $\Delta(X, \mathfrak{q}_i, \mathbb{F}_\ell)$ (where $\mathbb{F}_\ell$ is the finite field with ℓ elements).

Proposition 3.2.30. *Resume the notation of* (2.3.32) *and* (3.2.5)*. Then :*

$$2\alpha(\mathfrak{q}_i) + \mathfrak{F}(\mathfrak{q}_i) - 2 = \chi(\Delta(X, \mathfrak{q}_i, \mathbb{F}_\ell)[1]). \qquad \textit{for every } i = 1, \dots, n. \tag{3.2.31}$$

Proof. We shall give a global argument: first, according to [22, Ch.IV, Prop.17.7.8] and [43, Ch.XI, §2, Th.2], we can find :

- an étale map $\mathsf{A} \to \mathsf{A}'$ of K^+-algebras of finite presentation such that the induced map $\mathsf{A} \otimes_{K^+} K^\sim \to \mathsf{A}' \otimes_{K^+} K^\sim$ is an isomorphism;
- a finitely presented A'-algebra B with an isomorphism $\mathsf{A}^h \otimes_{\mathsf{A}'} \mathsf{B}' \xrightarrow{\sim} \mathsf{B}$. Especially, the induced morphism $\mathsf{A}' \otimes_{K^+} K \to \mathsf{B}' \otimes_{K^+} K$ is still étale.

Set $\mathsf{T} := \operatorname{Spec} \mathsf{A}'$ and $\mathsf{X} := \operatorname{Spec} \mathsf{B}'$. Using (3.2.1) one deduces natural isomorphisms in the derived category of complexes of $\mathbb{F}_\ell$-vector spaces:

$$\Delta(X, \mathfrak{q}_i, \mathbb{F}_\ell) \simeq R\Psi_{\mathsf{X}/S}(\mathbb{F}_{\ell,\mathsf{X}})_{\mathfrak{q}_i} \tag{3.2.32}$$

for every $i = 1, \dots, n$. The special fibre $\mathsf{X}_s := \operatorname{Spec} \mathsf{B}'/\mathfrak{m}\mathsf{B}'$ of the S-scheme X is of pure dimension one, since it is finite over $\operatorname{Spec} \mathsf{A}/\mathfrak{m}\mathsf{A}$, and it is reduced, in view of lemma 2.3.2. Hence X_s is generically smooth over $\operatorname{Spec} K^\sim$; denote by X_s^ν and $\mathsf{X}_s^{\mathrm{n}}$ respectively the seminormalization and normalization of X_s (cp. [39, §7.5, Def.13]). There are natural finite morphisms

$$\mathsf{X}_s^{\mathrm{n}} \xrightarrow{\pi_1} \mathsf{X}_s^\nu \xrightarrow{\pi_2} \mathsf{X}_s$$

and the quotient $\mathscr{O}_{\mathsf{X}_s}$-modules

$$Q_1 := (\pi_2 \circ \pi_1)_* \mathscr{O}_{\mathsf{X}_s^{\mathrm{n}}} / \pi_{2*} \mathscr{O}_{\mathsf{X}_s^\nu} \quad \text{and} \quad Q_2 := \pi_{2*} \mathscr{O}_{\mathsf{X}_s^\nu} / \mathscr{O}_{\mathsf{X}_s}$$

are torsion sheaves concentrated on the singular locus $\mathsf{X}_s^{\mathrm{sing}} \subset \mathsf{X}_s$. By inspecting the definition, one verifies easily that

$$\alpha(\mathfrak{q}_i) = \dim_{K^\sim} Q_{2,\mathfrak{q}_i} \qquad \text{for every } i = 1, \dots, n. \tag{3.2.33}$$

Similarly, using [41, Th.10.1] and lemma 2.2.12 one finds a natural bijection between the points of $\mathfrak{F}(\mathfrak{q}_i)$ and the points of $\mathsf{X}_s^{\mathrm{n}}$ lying over the point $\mathfrak{q}_i \in \mathsf{X}_s$ (cp. the proof of [31, Th.6.3]). This leads to the identity:

$$\sharp\mathfrak{F}(\mathfrak{q}_i) = 1 + \dim_{K^\sim} Q_{1,\mathfrak{q}_i} \qquad \text{for every } i = 1, \dots, n. \tag{3.2.34}$$

Now, let us fix one point $\mathfrak{q} := \mathfrak{q}_i$ and choose an affine open neighborhood $\mathsf{V} \subset \mathsf{X}$ of $\mathfrak{q}$ such that $\mathsf{X} \setminus \mathsf{V}$ contains $\mathsf{X}_s^{\mathrm{sing}} \setminus \{\mathfrak{q}\}$; by further restricting V we can even achieve that the special fibre V_s is connected. One can then apply theorem 3.1.3 to produce a projective S-scheme Y of pure relative dimension one containing an open subscheme U such that Y_s is connected, Y is smooth over S at the points of $\mathsf{Y}_s \setminus \mathsf{U}_s$, and furthermore $\mathsf{U}^h_{/\mathsf{U}_s} \simeq \mathsf{V}^h_{/\mathsf{V}_s}$. By construction, the generic fibre Y_K is a smooth projective curve over $\operatorname{Spec} K$, and $\mathsf{Y}_s^{\mathrm{sing}} \subset \{\mathfrak{q}\}$. It is also clear that the morphism $\mathsf{Y} \to S$ is flat, whence an equality of Euler-Poincaré characteristics:

$$\chi(\mathsf{Y}_K, \mathscr{O}_{\mathsf{Y}_K}) = \chi(\mathsf{Y}_s, \mathscr{O}_{\mathsf{Y}_s}).$$

On the other hand, let c be the number of irreducible components of Y_s; [39, §7.5, Cor.33] yields the identity

$$\begin{aligned}\dim_{\mathbb{F}_\ell} H^1(\mathsf{Y}_{s,\text{ét}}, \mathbb{F}_\ell) &= \dim_{\mathbb{F}_\ell} H^1(\mathsf{Y}^{\mathrm{n}}_{s,\text{ét}}, \mathbb{F}_\ell) + \dim_{K^\sim}(\pi_* \mathscr{O}_{\mathsf{Y}_s^{\mathrm{n}}} / \mathscr{O}_{\mathsf{Y}_s^\nu}) - c + 1 \\ &= \dim_{\mathbb{F}_\ell} H^1(\mathsf{Y}^{\mathrm{n}}_{s,\text{ét}}, \mathbb{F}_\ell) + \mathfrak{F}(\mathfrak{q}) - c\end{aligned}$$

where $\mathsf{Y}_s^{\mathrm{n}} \xrightarrow{\pi} \mathsf{Y}_s^\nu$ is the natural morphism from the normalization to the seminormalization of Y_s. Since $\dim_{\mathbb{F}_\ell} H^0(\mathsf{Y}^{\mathrm{n}}_{s,\text{ét}}, \mathbb{F}_\ell) = c$, we deduce

$$\chi_{\text{ét}}(\mathsf{Y}_s, \mathbb{F}_\ell) = \chi_{\text{ét}}(\mathsf{Y}_s^{\mathrm{n}}, \mathbb{F}_\ell) - \mathfrak{F}(\mathfrak{q}) + 1.$$

Furthermore, in light of (3.2.33) and (3.2.34) we can write

$$\chi(\mathsf{Y}_s, \mathscr{O}_{\mathsf{Y}_s}) = \chi(\mathsf{Y}_s^{\mathrm{n}}, \mathscr{O}_{\mathsf{Y}_s}) - \alpha(\mathfrak{q}) - \sharp\mathfrak{F}(\mathfrak{q}) + 1.$$

By Riemann-Roch we have

$$\chi_{\text{ét}}(\mathsf{Y}_s^{\mathrm{n}}, \mathbb{F}_\ell) = 2\chi(\mathsf{Y}_s^{\mathrm{n}}, \mathscr{O}_{\mathsf{Y}_s}) \qquad \text{and} \qquad \chi_{\text{ét}}(\mathsf{Y}_K, \mathbb{F}_\ell) = 2\chi(\mathsf{Y}_K, \mathscr{O}_{Y_s}).$$

Putting everything together we end up with the identity:

$$\chi_{\text{ét}}(\mathsf{Y}_K, \mathbb{F}_\ell) = \chi_{\text{ét}}(\mathsf{Y}_s, \mathbb{F}_\ell) - \sharp\mathfrak{F}(\mathfrak{q}) + 1 - 2\alpha(\mathfrak{q}).$$

Now, the complex $R\Phi_{\mathsf{Y}/S}(\mathbb{F}_\ell)$ is concentrated at $\mathsf{Y}_s^{\mathrm{sing}} \subset \{\mathfrak{q}\}$; it follows that $R\Phi_{\mathsf{Y}/S}(\mathbb{F}_\ell) \simeq j_! R\Phi_{\mathsf{U}/S}(\mathbb{F}_\ell)$, where $j : \mathsf{U}_s \to \mathsf{Y}_s$ is the open imbedding. Since furthermore the henselizations

of U and V are isomorphic, we have a natural identification $R\Phi_{\mathsf{U}/S}(\mathbb{F}_\ell)_\mathfrak{q} \simeq R\Phi_{\mathsf{V}/S}(\mathbb{F}_\ell)_\mathfrak{q} \simeq R\Phi_{\mathsf{X}/S}(\mathbb{F}_\ell)_\mathfrak{q}$. Consequently

$$\chi(R\Phi_{\mathsf{X}/S}(\mathbb{F}_\ell)_\mathfrak{q}) = \chi_{\text{ét}}(\mathsf{Y}_K, \mathbb{F}_\ell) - \chi_{\text{ét}}(\mathsf{Y}_s, \mathbb{F}_\ell) = 1 - \sharp\mathfrak{F}(\mathfrak{q}) - 2\alpha(\mathfrak{q}). \tag{3.2.35}$$

Clearly $\chi(R\Psi_{\mathsf{X}/S}(\mathbb{F}_\ell)_\mathfrak{q}) = 1 + \chi(R\Phi_{\mathsf{X}/S}(\mathbb{F}_\ell)_\mathfrak{q})$, whence the contention. □

3.3. **Conductors.** We consider now a finite Galois étale covering $f : X \to \mathbb{D}(a, b)$ of Galois group G. For given $r \in (a, b] \cap \Gamma_K$, pick any $x \in f^{-1}(\eta(r))$; G acts transitively on the set $f^{-1}(\eta(r))$ and the stabilizer subgroup $St_x \subset G$ of x is naturally isomorphic to the Galois group of the extension of henselian valued fields $\kappa(r)^{\wedge h} \subset \kappa(x)^{\wedge h}$ (see [31, §5.5]). By [31, Prop.1.2(iii) and Cor.5.4], $\kappa(x)^{\wedge h+}$ is a free $\kappa(r)^{\wedge h+}$-module of rank equal to the order $o(St_x)$ of St_x; hence the different $\mathscr{D}^+_{x/\eta(r)}$ of the ring extension $\kappa(r)^{\wedge h+} \subset \kappa(x)^{\wedge h+}$ is well-defined and, by arguing as in (2.1.7) one sees that it is principal. Moreover, it is shown in [31, Lemma 2.1(iii)] that for every $\sigma \in St_x$ there is a value $i_x(\sigma) \in \Gamma_x^+ \cup \{0\}$ such that

$$|t - \sigma(t)|_x^{\wedge h} = i_x(\sigma)$$

for all $t \in \kappa(x)^{\wedge h}$ such that $|t|_x^{\wedge h}$ is the largest element of $\Gamma_x^+ \setminus \{0\}$. The same argument as in the case of discrete valuations shows the identity

$$|\mathscr{D}^+_{x/\eta(r)}|_x^{\wedge h} = \prod_{\sigma \in St_x \setminus \{1\}} i_x(\sigma). \tag{3.3.1}$$

One defines the *higher ramification subgroups* of St_x by setting :

$$P_\gamma := \{\sigma \in St_x \mid i_x(\sigma) < \gamma\} \qquad \text{for every } \gamma \in \Gamma_x^+$$

and one says that $\gamma \in \Gamma_x^+$ is a *jump* in the family $(P_\gamma \mid \gamma \in \Gamma_x^+)$ if $P_{\gamma'} \neq P_\gamma$ for every $\gamma' < \gamma$. When $\gamma < 1$, the subgroup P_γ is contained in the unique p-Sylow subgroup $St_x^{(p)}$ of St_x.

Furthermore, one has Artin and Swan characters; to explain this, let us introduce the *total Artin conductor*:

$$\mathsf{a}_x : St_x \to \Gamma_x \qquad \mathsf{a}_x(\sigma) := \begin{cases} i_x(\sigma)^{-1} & \text{if } \sigma \neq 1 \\ \prod_{\tau \in St_x \setminus \{1\}} i_x(\tau) & \text{if } \sigma = 1. \end{cases}$$

It is convenient to decompose this total conductor into two (normalized) factors:

$$\mathsf{a}_x^\natural(\sigma) := o(St_x) \cdot \mathsf{a}_x(\sigma)^\natural \quad \text{and} \quad \mathsf{a}_x^\flat(\sigma) := -o(St_x) \cdot \log \mathsf{a}_x(\sigma)^\flat \qquad \text{for all } \sigma \in St_x.$$

and as usual the Swan character is $\mathsf{sw}_x^\natural := \mathsf{a}_x^\natural - \mathsf{u}_{St_x}$, where $\mathsf{u}_{St_x} := \mathsf{reg}_{St_x} - 1_{St_x}$ is the augmentation character, *i.e.* the regular character reg_{St_x} minus the constant function $1_{St_x}(\sigma) := 1$ for every $\sigma \in St_x$. Huber shows that $\mathsf{a}_x^\natural$ (resp. $\mathsf{sw}_x^\natural$) is the character of an element of $K_0(\mathbb{Q}_\ell[St_x])$ (resp. of $K_0(\mathbb{Z}_\ell[St_x])$: see [31, Th.4.1]); these are respectively the Artin and Swan representations. In general, these elements are however only virtual representations (whereas it is well known that in the case of discrete valuation rings one obtains actual representations).

3.3.2. The identities obtained in [47, §19.1] also generalize as follows. Let $\gamma_0 := 1 \in \Gamma_x$ and $\gamma_1 > \gamma_2 > \cdots > \gamma_n$ be the jumps in the family $(P_\gamma \mid \gamma \in \Gamma_x^+)$ which are < 1. Then directly from the definitions we deduce the identities :

$$\mathsf{a}_x^\flat = \sum_{i=1}^n o(P_{\gamma_i}) \cdot \log \frac{\gamma_{i-1}^\flat}{\gamma_i^\flat} \cdot \mathrm{Ind}_{P_{\gamma_i}}^{St_x} \mathsf{u}_{P_{\gamma_i}} \qquad \mathsf{sw}_x^\natural = \sum_{i=1}^n o(P_{\gamma_i}) \cdot (\gamma_i^\natural - \gamma_{i-1}^\natural) \cdot \mathrm{Ind}_{P_{\gamma_i}}^{St_x} \mathsf{u}_{P_{\gamma_i}} \tag{3.3.3}$$

where $\mathsf{u}_{P_{\gamma_i}}$ is the augmentation character of the group P_{γ_i}. Especially, notice that there exist two $\mathbb{R}$-valued and, respectively, $\mathbb{Q}$-valued class functions $\mathsf{a}^\flat_{St^{(p)}_x}$ and $\mathsf{sw}^\natural_{St^{(p)}_x}$ of the p-Sylow subgroup $St^{(p)}_x$, such that:

$$\mathsf{a}^\flat_x = \mathrm{Ind}^{St_x}_{St^{(p)}_x} \mathsf{a}^\flat_{St^{(p)}_x} \qquad \text{and} \qquad \mathsf{sw}^\natural_x = \mathrm{Ind}^{St_x}_{St^{(p)}_x} \mathsf{sw}^\natural_{St^{(p)}_x}$$

(the induced class functions from the subgroup $St^{(p)}_x$ to St_x; here we view $\mathsf{a}^\flat_x$ as an $\mathbb{R}$-valued class function). Next, we define

$$\mathsf{a}^\natural_G(r^+) := \mathrm{Ind}^G_{St_x} \mathsf{a}^\natural_x \qquad \mathsf{a}^\flat_G(r^+) := \mathrm{Ind}^G_{St^{(p)}_x} \mathsf{a}^\flat_{St^{(p)}_x} \qquad \mathsf{sw}^\natural_G(r^+) := \mathrm{Ind}^G_{St^{(p)}_x} \mathsf{sw}^\natural_{St^{(p)}_x}.$$

Notice that $\mathsf{a}^\natural_G(r^+)$ and $\mathsf{sw}^\natural_G(r^+)$ do not depend on the choice of the point $x \in f^{-1}(\eta(r))$.

Moreover, for any element $\chi \in K_0(\mathbb{C}[G])$ we let :

$$\mathsf{a}^\flat_G(\chi, r^+) := \langle \mathsf{a}^\flat_G(r^+), \chi \rangle_G \qquad \mathsf{sw}^\natural_G(\chi, r^+) := \langle \mathsf{sw}^\natural_G(r^+), \chi \rangle_G$$

where $\langle \cdot, \cdot \rangle_G$ is the natural scalar product of $\mathbb{R} \otimes_\mathbb{Z} K_0(\mathbb{C}[G])$ ([47, §7.2]). For future reference we point out :

Lemma 3.3.4. (i) $\mathsf{a}^\flat_G(\chi, r^+) \in \mathbb{R}$ *for every* $\chi \in K_0(\mathbb{C}[G])$.

(ii) *Moreover,* $\mathsf{a}^\flat_G(\chi, r^+) \geq 0$ *whenever* $\mathrm{Res}^G_{St^{(p)}_x} \chi$ *is a positive element of* $K_0(\mathbb{C}[St^{(p)}_x])$.

Proof. Both assertions follow easily from (3.3.3). □

3.3.5. Now, suppose that $f' : X' \to \mathbb{D}(a,b)$ is another finite Galois étale covering which dominates X, *i.e.* such that f' factors through f and an étale morphism $g : X' \to X$. Then $g(X')$ is a union of connected components of X. Let G' be the Galois group of f'; we assume as well that g is equivariant for the G'-action on X' and the G-action on X, *i.e.* there is a group homomorphism

$$\phi : G' = \mathrm{Aut}(X'/\mathbb{D}(a,b)) \to G = \mathrm{Aut}(X/\mathbb{D}(a,b))$$

such that $\phi(\sigma) \circ g = g \circ \sigma$ for every $\sigma \in G'$. Pick $x' \in X'$ lying over x; there follows a commutative diagram of group homomorphisms :

$$\begin{tikzcd} St_{x'} \arrow[r] \arrow[d] & St_x \arrow[d] \\ G' \arrow[r, "\phi"] & G \end{tikzcd}$$

whose vertical arrows are injections and whose top horizontal arrow is a surjection. One shows as in [46, Ch.VI, §2, Prop.3] that :

$$\mathsf{a}^\flat_x = \mathrm{Ind}^{St_x}_{St_{x'}} \mathsf{a}^\flat_{x'} \qquad \mathsf{a}^\natural_x = \mathrm{Ind}^{St_x}_{St_{x'}} \mathsf{a}^\natural_{x'}$$

whence the identities :

$$\mathsf{a}^\flat_G(r^+) = \mathrm{Ind}^G_{G'} \mathsf{a}^\flat_{G'}(r^+) \qquad \mathsf{a}^\natural_G(r^+) = \mathrm{Ind}^G_{G'} \mathsf{a}^\natural_{G'}(r^+). \tag{3.3.6}$$

3.3.7. Later we shall also need to know that the conductors are invariant under changes of base field. Namely, let $(K, |\cdot|) \to (F, |\cdot|_F)$ be a map of algebraically closed valued fields of rank one; say that $X = \mathrm{Spa}\, B$, and set :

$$X_F := \mathrm{Spa}\, B \widehat{\otimes}_K F \qquad \mathbb{D}(a,b)_F := \mathrm{Spa}\, A(a,b) \widehat{\otimes}_K F.$$

The natural map of adic space $X_F \to X$ is surjective and G-equivariant, hence we may choose $x' \in X_F$ lying over x. Let $f' : X_F \to \mathbb{D}(a,b)_F$ be the morphism deduced by base change from f. Then we may define Artin and Swan conductors for the extension $\kappa(f'(x'))^\wedge \subset \kappa(x')^\wedge$.

Lemma 3.3.8. *In the situation of* (3.3.7)*, the following holds :*

(i) *The natural map* $\kappa(x) \to \kappa(x')$ *induces isomorphisms on value groups and Galois groups :*

$$\Gamma_x \xrightarrow{\sim} \Gamma_{x'} \qquad St_{x'} \xrightarrow{\sim} St_x. \tag{3.3.9}$$

(ii) *Under the identification* (3.3.9), *we have :*

$$i_x(\sigma) = i_{x'}(\sigma) \qquad \textit{for every } \sigma \in St_x.$$

Proof. Set $\eta(r)_F := f'(x')$ and $\kappa(r)_F := \kappa(\eta(r)_F)$; one checks easily that the natural commutative diagram :

$$\begin{CD}
\kappa(r) @>>> \mathscr{B}(r) := (f_*\mathscr{O}_X)_{\eta(r)} \\
@VVV @VVV \\
\kappa(r)_F @>>> \mathscr{B}(r)_F := (f'_*\mathscr{O}_{X_F})_{\eta(r)_F}
\end{CD}$$

is cocartesian. By lemma 2.2.12, the set $f^{-1}(\eta(r))$ (resp. $f'^{-1}(\eta(r)_F)$) is in natural bijection with the set of valuations on $\mathscr{B}(r)$ (resp. $\mathscr{B}(r)_F$) that extend $|\cdot|_{\eta(r)}$ (resp. $|\cdot|_{\eta(r)_F}$). It then follows by standard valuation theory (see [9, Ch.VI, §2, Exerc.2]) that the map $f'^{-1}(\eta(r)_F) \to f^{-1}(\eta(r))$ is a surjection; let N (resp. N') be the cardinality of $f^{-1}(\eta(r))$ (resp. $f'^{-1}(\eta(r)_F)$). Moreover, $\mathscr{B}(r)_F$ is reduced, by [11, Lemma 3.3.1.(1)]; let r be its rank over $\kappa(r)_F$, which is also the rank of $\mathscr{B}(r)$ over $\kappa(r)$. Since $\kappa(\eta(r))$ and $\kappa(\eta(r)_F)$ are defectless in every finite separable extension ([31, Lemma 5.3(iii)]), we deduce that :

$$r = N \cdot (\Gamma_x : \Gamma_{\eta(r)}) = N' \cdot (\Gamma_{x'} : \Gamma_{\eta(r)_F}).$$

We know already that $N' \geq N$, and since $\Gamma_{\eta(r)} = \Gamma_{\eta(r)_F}$, it is also clear that $(\Gamma_{x'} : \Gamma_{\eta(r)_F}) \geq (\Gamma_x : \Gamma_{\eta(r)})$, whence (i). Next, in light of (i), for every $\sigma \in St_{x'}$ we may compute $i_{x'}(\sigma)$ as $|\sigma(t) - t|_{x'}$, where $t \in \kappa(x)$ is any element such that $|t|_x$ is the largest element in $\Gamma_x^+ \setminus \{1\}$. Assertion (ii) is then an immediate consequence. □

Lemma 3.3.10. *For every subgroup* $H \subset G$, *denote by* $f_H : X/H \to \mathbb{D}(a, a^{-1})$ *the morphism deduced from* f. *The following identities hold:*

$$\delta_{f_H}(-\log r) = \mathsf{a}^\flat_G(\mathbb{C}[G/H], r^+) \qquad \textit{for every } r \in (a, b] \cap \Gamma_K$$

and

$$\frac{d\delta_{f_H}}{dt}(-\log r^+) = \mathsf{sw}^\natural_G(\mathbb{C}[G/H], r^+) \qquad \textit{for every } r \in (a, b] \cap \Gamma_K.$$

Proof. (Here $\mathbb{C}[G/H] = \mathrm{Ind}_H^G 1_H$, where 1_H is the trivial character of H.) First of all, one applies (3.3.1) to derive, as in [46, Ch.VI, §3, Cor.1] that

$$|\mathfrak{d}^+_{f_H}(r)|^\natural_{\eta(r)} = \langle \mathsf{a}^\natural_G(r^+), \mathrm{Ind}_H^G 1_H \rangle_G \qquad \text{and} \qquad -\log|\mathfrak{d}^+_{f_H}(r)|^\flat_{\eta(r)} = \langle \mathsf{a}^\flat_G(r^+), \mathrm{Ind}_H^G 1_H \rangle_G$$

(notation of (2.3.12)), which already implies the first of the sought identities. Moreover we deduce:

$$\begin{aligned}
\langle \mathsf{sw}^\natural_G(r^+), \mathrm{Ind}_H^G 1_H \rangle_G &= |\mathfrak{d}^+_{f_H}(r)|^\natural_{\eta(r)} - \langle \mathrm{Ind}^G_{St_x} \mathsf{u}_{St_x}, \mathrm{Ind}_H^G 1_H \rangle_G \\
&= |\mathfrak{d}^+_{f_H}(r)|^\natural_{\eta(r)} + \sharp(H\backslash G/St_x) - (G : H).
\end{aligned}$$

which is equivalent to the second stated identity, in view of lemma 2.2.17(iii) and proposition 2.3.17. □

3.3.11. Of course, one can also repeat the same discussion with the point $\eta'(r)$ instead of $\eta(r)$ (notation of (2.2.10)); then one obtains characters $\mathsf{a}^\natural_G(r^-)$, $\mathsf{a}^\flat_G(r^-)$, $\mathsf{sw}^\natural_G(r^-)$ and – in view of example (2.3.16) – the identity:

$$-\frac{d\delta_f}{dt}(-\log r^-) = \langle \mathsf{sw}^\natural_G(r^-), \mathsf{reg}_G\rangle_G \qquad \text{for every } r \in [a,b) \cap \Gamma_K. \tag{3.3.12}$$

Moreover we have :

Lemma 3.3.13. $\mathsf{a}^\flat_G(r^-) = \mathsf{a}^\flat_G(r^+)$.

Proof. In view of (3.3.3) and its analogue for $\mathsf{a}^\flat_G(r^-)$, we see that both sides of the sought identity are elements of $\mathbb{R} \otimes_\mathbb{Z} R_\mathbb{Q}(G)$ (notation of [47, §12.1]). By [47, §13.1, Th.30], it then suffices to check that $\mathsf{a}^\flat_G(\mathbb{C}[G/H], r^-) = \mathsf{a}^\flat_G(\mathbb{C}[G/H], r^+)$ for every (cyclic) subgroup $H \subset G$. The latter is clear, in light of lemma 3.3.10. □

Proposition 3.3.14. *For every $\chi \in K_0(\mathbb{C}[G])$ the function*

$$\delta_{f,\chi} : (\log 1/b, \log 1/a] \cap \log \Gamma_K \to \mathbb{R} \quad : \quad -\log r \mapsto \mathsf{a}^\flat_G(\chi, r^+)$$

is the restriction of a piecewise linear continuous function defined on $(\log 1/b, \log 1/a]$*, and the following identity holds :*

$$\frac{d\delta_{f,\chi}}{dt}(-\log r^+) = \mathsf{sw}^\natural_G(\chi, r^+) \qquad \textit{for every } r \in (a,b] \cap \Gamma_K.$$

Proof. Due to Artin's theorem [47, §9.2, Cor.] we may assume that χ is induced from a character $\rho : H \to \mathbb{C}^\times$ of a cyclic subgroup $H \subset G$. Denote by $\mathsf{a}^\flat_{|H}$ (resp. $\mathsf{sw}^\natural_{|H}$) the restriction to H of $\mathsf{a}^\flat_G$ (resp. of $\mathsf{sw}^\natural_G$); by Frobenius reciprocity, we are reduced to showing that the map :

$$-\log r \mapsto \langle \mathsf{a}^\flat_{|H}(r^+), \rho\rangle_H$$

extends to a piecewise linear and continuous function, with right slope $\langle \mathsf{a}^\natural_{|H}(r^+), \rho\rangle_H$. Let k be the order of ρ, *i.e.* the smallest integer such that $\rho^k = 1_H$; we shall argue by induction on k. For $k = 1$, ρ is the trivial character, and then the assertion follows from lemma 3.3.10. Hence, suppose that $k > 1$ and that the assertion is known for all the characters of H whose order is strictly smaller than k. There exists a unique subgroup $L \subset H$ with $(H : L) = k$, and $\mathbb{C}[H/L] \subset \mathbb{C}[H]$ is the direct sum of all characters of H whose orders divide k. By lemma 3.3.10 the sought assertion is known for this direct sum of characters, and then our inductive assumption implies that the assertion is also known for the sum $\rho' := \rho_1 \oplus \cdots \oplus \rho_n$ of all characters of H whose order equals k. The latter are permuted under the natural action of $(\mathbb{Z}/k\mathbb{Z})^\times$ on $K_0(\mathbb{C}[H])$ (cp. [47, §9.1, Exerc.3]). Moreover, for any $j \in (\mathbb{Z}/k\mathbb{Z})^\times$ let $\Psi_j : K_0(\mathbb{C}[H]) \to K_0(\mathbb{C}[H])$ be the corresponding operator.

Claim 3.3.15. $\mathsf{a}^\flat_{|H} = \Psi_j(\mathsf{a}^\flat_{|H})$ for every $j \in (\mathbb{Z}/k\mathbb{Z})^\times$.

Proof of the claim. From [31, Lemma 2.6] it follows that $\mathsf{a}_x(\sigma) = \mathsf{a}_x(\tau)$ whenever σ and τ generate the same subgroup of St_x. The claim is a direct consequence. ◊

Using claim 3.3.15 we compute :

$$\langle \mathsf{a}^\flat_{|H}(r^+), \rho\rangle = \langle \Psi_j(\mathsf{a}^\flat_{|H}(r^+)), \Psi_j(\rho)\rangle = \langle \mathsf{a}^\flat_{|H}(r^+), \Psi_j(\rho)\rangle$$

for every $r \in (a,b] \cap \Gamma_K$ and every $j \in (\mathbb{Z}/k\mathbb{Z})^\times$. Thus :

$$\mathsf{a}^\flat_G(\chi, r^+) = \frac{\mathsf{a}^\flat_G(\rho, r^+)}{o((\mathbb{Z}/k\mathbb{Z})^\times)}.$$

A similar argument yields a corresponding identity for $\mathsf{sw}^\natural_G(\chi, r^+)$, and concludes the proof of the proposition. □

The following is the main result of this chapter.

Theorem 3.3.16. *Suppose that $b = a^{-1}$, so that we can define the complex of $\mathbb{Z}_\ell[G]$-modules $\Delta^\bullet_\infty(X)$ as in* (3.2.12)*. Then we have the identity :*

$$[\mathbb{Q} \otimes_\mathbb{Z} \Delta^\bullet_\infty(X)[1]] = \mathsf{sw}^\natural_G(1^+) + \mathsf{sw}^\natural_G(1^-).$$

Proof. We begin with the following:

Claim 3.3.17. For every abelian subgroup $H \subset G$ we have a natural identification:

$$\Delta^\bullet_\infty(X)^H = \Delta^\bullet_\infty(X/H).$$

Proof of the claim. Indeed, let $\mathfrak{p}_1,\dots,\mathfrak{p}_k$ be the prime ideals of $(B^\circ)^H$ lying over $\mathfrak{P}$ (notation of (3.2.5)); for every $j := 1, \dots, k$ let S_j be the set of the prime ideals of B° lying over $\mathfrak{p}_j$. Clearly $\bigcup_{j=1}^k S_j = \{\mathfrak{q}_1, \dots, \mathfrak{q}_n\}$, and for every $j \leq k$ the subgroup H stabilizes the direct sum

$$\Delta^\bullet_\infty(X, S_j) := \bigoplus_{\mathfrak{q}\in S_j} \Delta^\bullet_\infty(X, \mathfrak{q}).$$

Let $L_j \subset H$ be the stabilizer of any (hence each) element of S_j; by lemma 3.2.27 we have a natural identification:

$$\Delta^\bullet_\infty(X, S_j)^{L_j} = \bigoplus_{\mathfrak{q}\in S_j} \Delta^\bullet_\infty(X/L_j, \mathfrak{q}^{L_j}) = \bigoplus_{\mathfrak{q}\in S_j} \Delta^\bullet_\infty(X/H, \mathfrak{p}_j).$$

However, the quotient H/L_j permutes transitively the summands of $\Delta^\bullet_\infty(X, S_j)^{L_j}$, whence the claim. $\diamondsuit$

Claim 3.3.18. For every cyclic subgroup $H \subset G$ we have:

$$\langle[\mathbb{Q} \otimes_\mathbb{Z} \Delta^\bullet_\infty(X)[1]], \mathbb{C}[G/H]\rangle_G = \langle\mathsf{sw}^\natural_G(1^+) + \mathsf{sw}^\natural_G(1^-), \mathbb{C}[G/H]\rangle_G. \tag{3.3.19}$$

Proof of the claim. We use the morphism $h := \mathrm{Spa}\,\psi \circ f$ defined in example 2.3.13. For $H \subset G$ any cyclic subgroup, set $h_H := \mathrm{Spa}\,\psi \circ f_H$. In view of proposition 3.2.30 and claim 3.3.17, and arguing as in the proof of theorem 2.3.35, we see that the left-hand side of (3.3.19) computes (-1) times the left slope at the point 0 of the discriminant function of h_H. But combining lemma 3.3.10, (3.3.12) and example 2.3.16 we conclude that also the right-hand side admits the same interpretation. $\diamondsuit$

Since $\mathsf{sw}^\sharp_G(1^+) + \mathsf{sw}^\sharp_G(1^-)$ is a rational valued function on G, the theorem follows from claim 3.3.18, theorem 3.2.17(ii) and [47, §13.1, Th.30]. $\square$

Corollary 3.3.20. *Keep the notation of proposition* 3.3.14*. For every complex representation ρ of G, the function $\delta_{f,\rho}$ is the restriction of a non-negative, piecewise linear, continuous, and convex real-valued function defined on $(\log 1/b, \log 1/a]$, with integer slopes.*

Proof. Continuity and piecewise linearity are already known from proposition 3.3.14, and $\delta_{f,\rho}$ takes values in $\mathbb{R}_{\geq 0}$, by lemma 3.3.4(i),(ii). In view of [31, Th.4.1], proposition 3.3.14 also implies that the slopes of $\delta_{f,\rho}$ are integer. Finally, let $r \in (a, b] \cap \Gamma_K$ be a radius such that the left and right slope of $\delta_{f,\rho}$ are different at the value $-\log r$; up to restricting the covering f, and rescaling the coordinate, we may assume that $r = 1$ and $b = a^{-1}$, in which case theorem 3.3.16, lemma 3.3.13 and (3.3.12) yield the identity :

$$\langle[\mathbb{Q} \otimes_\mathbb{Z} \Delta^\bullet_\infty(X)][1], \rho\rangle_G = \frac{d\delta_{f,\rho}}{dt}(0^+) - \frac{d\delta_{f,\rho}}{dt}(0^-). \tag{3.3.21}$$

However, lemma 3.2.15 implies that the left-hand side of (3.3.21) is a non-negative integer for any representation ρ, whence the contention. $\square$

3.3.22. Next we consider modular representations of G. Namely, let Λ be a complete discrete valuation ring with residue field $\overline{\Lambda}$ of positive characteristic $\ell \neq p$, and field of fractions $\Lambda_{\mathbb{Q}} := \Lambda[1/\ell]$ of characteristic zero. We assume that we are also given a fixed imbedding of $\Lambda_{\mathbb{Q}}$ into the field of complex numbers :

$$\Lambda_{\mathbb{Q}} \hookrightarrow \mathbb{C}. \tag{3.3.23}$$

As usual (cp. (3.2.9)), for any group H we denote by $K_0(\overline{\Lambda}[H])$ (resp. by $K^0(\overline{\Lambda}[H])$) the Grothendieck group of the category of $\overline{\Lambda}[H]$-modules of finite rank over $\overline{\Lambda}$ (resp. of projective $\overline{\Lambda}[H]$-modules of finite rank). We shall also consider $K_0(\Lambda_{\mathbb{Q}}[H]) = K^0(\Lambda_Q[H])$. The tensor product (over Λ) induces a ring structure on these groups, and according to [47, §15.5, Prop.43] there is a commutative diagram of ring homomorphisms :

$$\begin{tikzcd} K^0(\overline{\Lambda}[H]) \arrow[rr, "c_H"] \arrow[dr, "e_H"'] & & K_0(\overline{\Lambda}[H]) \\ & K_0(\Lambda_{\mathbb{Q}}[H]) \arrow[ur, "d_H"'] & \end{tikzcd}$$

such that d_H and e_H are adjoint maps for the natural bilinear pairing

$$\langle\cdot,\cdot\rangle_H : K^0(\overline{\Lambda}[H]) \times K_0(\overline{\Lambda}[H]) \to \mathbb{Z}$$

i.e. we have the identity ([47, §15.4]) :

$$\langle \rho, d_H(\chi)\rangle_H = \langle e_H(\rho), \chi\rangle_H \qquad \text{for every } \chi \in K_0(\Lambda_{\mathbb{Q}}[H]) \text{ and } \rho \in K^0(\overline{\Lambda}[H]). \tag{3.3.24}$$

3.3.25. For instance, for every $r \in (a,b] \cap \Gamma_K$ there exists a unique element $\overline{\mathsf{sw}}^\natural_G(r^+) \in K^0(\overline{\Lambda}[G])$ such that $\mathsf{sw}^\natural_G(r^+) = e_G(\overline{\mathsf{sw}}^\natural_G(r^+))$. Likewise, by inspecting (3.3.3) we see that there exist elements

$$\overline{\mathsf{a}}^\flat_G(r^+) \in \mathbb{R} \otimes_{\mathbb{Z}} K^0(\overline{\Lambda}[G]) \qquad \overline{\mathsf{a}}^\flat_{St^{(p)}_x}(r^+) \in \mathbb{R} \otimes_{\mathbb{Z}} K^0(\overline{\Lambda}[St^{(p)}_x])$$

such that :

$$\mathsf{a}^\flat_{St^{(p)}_x}(r^+) = e_{St^{(p)}_x}(\overline{\mathsf{a}}^\flat_{St^{(p)}_x}(r^+)) \qquad \mathsf{a}^\flat_G(r^+) = e_G(\overline{\mathsf{a}}^\flat_G(r^+)) \qquad \overline{\mathsf{a}}^\flat_G(r^+) = \mathrm{Ind}^G_{St^{(p)}_x} \overline{\mathsf{a}}^\flat_{St^{(p)}_x}(r^+).$$

Now, let $\overline{\chi} \in K_0(\overline{\Lambda}[G])$ be any element; we define the function :

$$\delta_{f,\overline{\chi}} : (\log 1/b, \log 1/a] \cap \log\Gamma_K \to \mathbb{C} \qquad -\log r \mapsto \langle \overline{\mathsf{a}}^\flat_G(r^+), \overline{\chi}\rangle_G.$$

Proposition 3.3.26. *If $\overline{\chi}$ is the class of a $\overline{\Lambda}$-linear representation (*i.e.* a positive element of $K_0(\overline{\Lambda}[G])$), then $\delta_{f,\overline{\chi}}$ is the restriction of a non-negative, piecewise linear, convex and continuous real-valued function defined on* $(\log 1/b, \log 1/a]$*, and moreover :*

$$\frac{d\delta_{f,\overline{\chi}}}{dt}(-\log r^+) = \langle \overline{\mathsf{sw}}^\natural_G(r^+), \overline{\chi}\rangle_G \tag{3.3.27}$$

Proof. According to [47, §16.1, Th.33], we may find $\chi \in K_0(\Lambda_{\mathbb{Q}}[G]) \subset K_0(\mathbb{C}[G])$ such that $d_G(\chi) = \overline{\chi}$. For every $r \in (a,b] \cap \Gamma_K$, we compute :

$$\delta_{f,\overline{\chi}}(-\log r) = \langle \overline{\mathsf{a}}^\flat_G(r^+), d_G(\chi)\rangle_G = \langle \mathsf{a}^\flat_G(r^+), \chi\rangle_G$$

and then piecewise linearity, continuity, as well as (3.3.27) follow from proposition 3.3.14. Since $\mathrm{Res}^G_{St^{(p)}_x}(\chi) = d^{-1}_{St^{(p)}_x} \circ \mathrm{Res}^G_{St^{(p)}_x}(\overline{\chi})$ is a positive element of $K_0(\mathbb{C}[St^{(p)}_x])$, lemma 3.3.4(i),(ii) implies that $\delta_{f,\overline{\chi}}(-\log r) \in [0,+\infty)$.

As for the convexity, notice that we cannot apply directly corollary 3.3.20, since the element $\chi \in K_0(\mathbb{C}[G])$ may fail to be positive. Nevertheless, the argument proceeds along the same lines : let $r \in (a,b] \cap \Gamma_K$ be a radius such that the left and right slopes of $\delta_{f,\overline{\chi}}$ are different at the

value $-\log r$; we may assume that $b = a^{-1}$ and $r = 1$, in which case the class $[\Delta^\bullet_\infty(X)[1]] \in K^0(\mathbb{Z}_\ell[G])$ is well defined and positive. Clearly we have :

$$[\mathbb{Q} \otimes_\mathbb{Z} \Delta^\bullet_\infty(X)[1]] = e_G([\Delta^\bullet(X, \overline{\Delta})[1]])$$

(notation of (3.2.5)). Then we compute using theorem 3.3.16, lemma 3.3.13 and (3.3.12) :

$$\frac{d\delta_{f,\overline{x}}}{dt}(0^+) - \frac{d\delta_{f,\overline{x}}}{dt}(0^-) = \langle [\mathbb{Q} \otimes_\mathbb{Z} \Delta^\bullet_\infty(X)][1], \chi\rangle_G = \langle [\Delta^\bullet(X, \overline{\Delta})[1]], \overline{\chi}\rangle_G \qquad (3.3.28)$$

where again the last identity follows from (3.3.24). To conclude we apply proposition 3.2.10(iii) to deduce the sought positivity of the right-most term in (3.3.28). □

Next, we wish to investigate the continuity properties of the higher ramification filtration. These are gathered in the following :

Theorem 3.3.29. *In the situation of* (3.3)*, there exists $r' \in (a, r)$ and, for every $s \in (r', r] \cap \Gamma_K$, a point $x(s) \in f^{-1}(\eta(s))$ such that :*

(i) *The stabilizer $St_{x(s)} \subset G$ of $x(s)$ under the natural G-action on $f^{-1}(\eta(s))$, is a subgroup independent of s.*
(ii) *The length of the higher ramification filtration of $St_{x(s)} = \mathrm{Gal}(\kappa(x(s))^{\wedge h}/\kappa(s)^{\wedge h})$:*

$$P_{\gamma_n(s)} \subset \cdots \subset P_{\gamma_1(s)} \subset St^{(p)}_{x(s)}$$

is independent of $s \in (r', r] \cap \Gamma_K$.
(iii) *Set $\gamma_k^\natural := \gamma_k(r)^\natural$ for every $k \leq n$. Then :*

$$\gamma_k(s) = (s/r)^{\gamma_k^\natural} \cdot \gamma_k(r) \qquad \textit{for every } s \in (r', r] \cap \Gamma_K.$$

Proof. Choose $r' \in (a, r) \cap \Gamma_K$ such that the discriminant function δ_f has constant left slope on $[r', r] \cap \Gamma_K$. Up to rescaling the coordinates, we may assume that $r = a$ and $r' = a^{-1}$ with $a := |\pi|$ for some $\pi \in \mathfrak{m}$. Then $\mathbb{D}(r', r) = \mathbb{D}(a, a^{-1}) = \mathrm{Spa}\, A(a, a^{-1})$, and $A^\circ := A(a, a^{-1})^\circ = K^+\langle S, T\rangle/(S \cdot T - \pi^2)$. Also, there is no harm in replacing X by its restriction to $f^{-1}(\mathbb{D}(r', r))$. Let B be an affinoid K-algebra such that $X = \mathrm{Spa}\, B$, and denote by $\mathfrak{P} \subset A^\circ$ the unique maximal ideal such that $S, T \in \mathfrak{P}$. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_k \subset B^\circ$ be the maximal ideals lying over $\mathfrak{P}$.

Claim 3.3.30. For every $s \in (a, a^{-1}] \cap \Gamma_K$ and every $i \leq k$ there is exactly one point $x_i(s) \in f^{-1}(\eta(s))$ such that $\kappa(x_i(s))^+$ dominates $B^\circ_{\mathfrak{q}_i}$.

Proof of the claim. One argues as in the proof of theorem 2.4.3, using (2.4.4) : the details shall be left to the reader. ◊

By standard arguments we may find an étale ring homomorphism $\phi : A^\circ \to C$ such that :

- $\mathrm{Spec}\, C$ is connected, $\mathfrak{P}' := \mathfrak{P}C$ is a maximal ideal of C, and the induced map $A^\circ/\mathfrak{P} \to C/\mathfrak{P}'$ is an isomorphism.
- $\mathrm{Spec}\, C \otimes_{A^\circ} B^\circ$ is the union of k connected components, which are therefore in natural bijection with the set $\{\mathfrak{q}_1, \dots, \mathfrak{q}_k\}$.

Let $C^\wedge$ be the π-adic completion of C; then $C^\wedge_K := C^\wedge \otimes_{K^+} K$ is an affinoid K-algebra, and the induced morphism of affinoid spaces $g : Y := \mathrm{Spa}\, C^\wedge_K \to \mathbb{D}(a, a^{-1})$ is étale ([30, Cor.1.7.3(iii)]).

Claim 3.3.31. (i) $(C^\wedge_K)^\circ = C^\wedge$.

(ii) g restricts to an isomorphism $Y_t := g^{-1}(\mathbb{D}(a/|t|, |t|/a)) \xrightarrow{\sim} \mathbb{D}(a/|t|, |t|/a)$, for every $t \in \mathfrak{m} \setminus \{0\}$.
(iii) For every $s \in (a, a^{-1}] \cap \Gamma_K$, the preimage $g^{-1}(\eta(s))$ consists of one point.

Proof of the claim. (i): Since C is K^+-flat, the same holds for $C^\wedge$ (cp. the proof of lemma 1.3.6); moreover $C/\mathfrak{m}C$ is reduced, since it is étale over the reduced ring $A^\sim$ (notation of (2.2.4)). Then the assertion follows from lemma 2.3.2.

Next, denote by $A^{\circ h}_{\mathfrak{P}}$ (resp. $C^h_{\mathfrak{P}'}$) the henselization of A° (resp. of C) along $\mathfrak{P}$ (resp. along $\mathfrak{P}'$); for every $t \in \mathfrak{m} \setminus \{0\}$, the map $A^\circ \to A^\circ_t := A(a/|t|, |t|/a)^\circ$ (dual to the open immersion $\mathbb{D}(a/|t|, |t|/a) \subset \mathbb{D}(a, a^{-1})$) factors through a natural morphism $A^{\circ h}_{\mathfrak{P}} \to A^\circ_t$. Since ϕ induces an isomorphism $A^{\circ h}_{\mathfrak{P}} \xrightarrow{\sim} C^h_{\mathfrak{P}'}$, assertion (ii) follows easily. This implies already assertion (iii), for every $s \in (a, a^{-1}) \cap \Gamma_K$. It remains to show that there exists exactly one point in $y \in Y$ lying over $\eta(a^{-1})$; in view of (i), this is the same as showing that $C \otimes_{K^+} K$ admits exactly one continuous valuation $|\cdot|_y : C \otimes_{K^+} K \to \Gamma_y \cup \{0\}$ such that

$$|t|_y = |t|_{\eta(a^{-1})} \quad \text{for every } t \in A^\circ \qquad \text{and} \qquad |t|_y \leq 1 \quad \text{for every } t \in C.$$

These valuations are in natural bijection with the set of valuations $|\cdot|_{\overline{y}}$ on $C/\mathfrak{m}C$ such that $|\overline{t}|_{\overline{y}} = |t|^\natural_{\eta(a^{-1})}$ for the class $\overline{t} \in A^\sim$ of every element $t \in A^\circ \setminus \mathfrak{m}A^\circ$. But since $C/\mathfrak{m}C$ is étale over $A^\sim$ and $\mathfrak{P}'$ is a prime ideal, there is exactly one such valuation $|\cdot|_{\overline{y}}$. ◊

Let $Z := X \times_{\mathbb{D}(a,a^{-1})} Y$; then $Z = \mathrm{Spa}\, C^\wedge_K \otimes_{A^\circ} B^\circ$. Set $D := C^\wedge \otimes_{A^\circ} B^\circ$.

Claim 3.3.32. $\mathscr{O}^+_Z(Z) = D$ and D is henselian along its ideal $\mathfrak{m}D$.

Proof of the claim. Obviously the second assertion follows from the first. Since B° is a finitely presented A°-module (lemma 2.3.1), D is the π-adic completion of $C \otimes_{A^\circ} B^\circ$, and since the latter is flat over K^+, the same holds for D (cp. the proof of lemma 1.3.6). Furthermore, $D/\mathfrak{m}D = (C/\mathfrak{m}C) \otimes_{A^\sim} B^\sim$. By lemma 2.3.2, the ring $B^\sim$ is reduced, hence the same holds for $D/\mathfrak{m}D$, and then the claim follows by a second application of lemma 2.3.2. ◊

It follows from claim 3.3.32 and the construction of C, that Z splits as the disjoint union of k open and closed subsets $Z_{\mathfrak{q}_i}$ ($i = 1, \dots, k$), in natural bijection with the set $\Sigma := \{\mathfrak{q}_1, \dots, \mathfrak{q}_k\}$, and such that precisely one prime ideal of $\mathscr{O}^+_Z(Z_{\mathfrak{q}_i})$ lies over $\mathfrak{P}'$, for every $i \leq k$. Choose arbitrarily $\mathfrak{q} \in \Sigma$, and let $Z_\mathfrak{q} \subset Z$ be the corresponding open and closed subset. The restriction $h : Z_\mathfrak{q} \to Y$ of $f \times_{\mathbb{D}(a,a^{-1})} Y$ is a finite Galois étale morphism, whose Galois group is the stabilizer $St(\mathfrak{q}) \subset G$ of $\mathfrak{q}$ for the natural G-action on Σ. Furthermore, combining claims 3.3.30, 3.3.31 we deduce that, for every $s \in (a, a^{-1}] \cap \Gamma_K$ there exists precisely one point $z(s) \in Z_\mathfrak{q}$ such that $\{h(z(s))\} = g^{-1}(\eta(s))$; set $y(s) := h(z(s))$ for every such s. Let $g' : Z_\mathfrak{q} \to X$ be the restriction of $X \times_{\mathbb{D}(a,b)} g$, and set $x(s) := g'(z(s))$ for every $s \in (a, a^{-1}] \cap \Gamma_K$. Since g' is equivariant for the action of $St(\mathfrak{q})$, we see that this family $(x(s) \mid s \in (a, a^{-1}] \cap \Gamma_K)$ fulfills condition (i); indeed we have $St_{x(s)} = St(\mathfrak{q})$ for each point $x(s)$.

Claim 3.3.33. For every $s \in (a, a^{-1}] \cap \Gamma_K$, the following holds :

(i) The natural maps $\kappa(s) \to \kappa(y(s))$ and $\kappa(x(s)) \to \kappa(z(s))$ induce $St(\mathfrak{q})$-equivariant identifications :

$$\kappa(s)^{\wedge h+} \xrightarrow{\sim} \kappa(y(s))^{\wedge h+} \qquad \text{and} \qquad \kappa(x(s))^{\wedge h+} \xrightarrow{\sim} \kappa(z(s))^{\wedge h+}$$

(ii) The map $\kappa(y(s)) \to \kappa(z(s))$ is a finite field extension, and $\kappa(z(s)) = (h_* \mathscr{O}_{Z_\mathfrak{q}})_{y(s)}$.

Proof of the claim. (i): For $s < a^{-1}$ this is already clear from claim 3.3.31(ii); indeed in this case one does not need to complete, nor to henselize, to obtain isomorphisms. For the case where $s = a^{-1}$, set $D := \kappa(a^{-1})^{\wedge h+} \otimes_{A^\circ} C$, and let $D^\nu \subset D$ be the normalization of $\kappa(a^{-1})^{\wedge h+}$ in D. Then $\mathrm{Spec}\, D$ and $\mathrm{Spec}\, D^\nu$ have the same number of irreducible components; moreover D^ν is a product of finitely many valuation rings, each of which is a finite extension of $\kappa(a^{-1})^{\wedge h+}$. Since C admits only one valuation extending $|\cdot|_{\eta(a^{-1})}$, we deduce that precisely one of the irreducible components of $\mathrm{Spec}\, D$ is finite over $\kappa(a^{-1})^{\wedge h+}$. Let V be the direct factor of D corresponding to this irreducible component; then $V \cap \mathrm{Frac}(C)$ is the valuation ring of the field of fractions

$\mathrm{Frac}(C)$, corresponding to $y(a^{-1}) \in Y$. On the one hand, the extension $\kappa(a^{-1})^{\wedge h+} \to V$ is étale, hence an isomorphism; on the other hand, the natural map $V \to \kappa(y(a^{-1}))^{\wedge h+}$ induces a bijection on value groups, so the map $\kappa(a^{-1})^{\wedge h+} \to \kappa(y(a^{-1}))^{\wedge h+}$ is an isomorphism, as stated. The remaining assertion concerning $\kappa(x(a^{-1}))^{\wedge h+}$ is an easy consequence; the details shall be left to the reader.

(ii): It suffices to show that $z(s)$ admits a cofinal system of open neighborhoods $U \subset Z_{\mathfrak{q}}$ such that the restriction $h_{|U} : U \to h(U)$ is finite. However, let $V \subset Z_{\mathfrak{q}}$ be any open neighborhood of $z(s)$; since $\{z(s)\} = h^{-1}(h(z(s)))$, the subset $U := h^{-1}(Y \setminus h(Z_{\mathfrak{q}} \setminus V)) \subset V$ is also an open neighborhood of $z(s)$, and clearly $h_{|U}$ is finite. $\Diamond$

Claim 3.3.34. Let $U \subset Z_{\mathfrak{q}}$ be an open neighborhood of $z(a^{-1})$, $t \in \mathscr{O}_Z(U)$ any section, and set $k := |t|^{\natural}_{z(a^{-1})}$. Then there exists $r \in (a, a^{-1})$ such that :

(i) $z(s) \in U$ for every $s \in [r, a^{-1}] \cap \Gamma_K$.
(ii) $|t|_{z(s)} = |t|_{z(a^{-1})} \cdot (s \cdot a)^k$ for every $s \in [r, a^{-1}] \cap \Gamma_K$.

Proof of the claim. Notice that claim 3.3.33(ii), and lemma 1.1.17(iii) imply that the spectral norm induced on the $\kappa(y(s))$-algebra $\kappa(z(s))$ agrees with the valuation $|\cdot|_{z(s)}$, for every $s \in (a, a^{-1}] \cap \Gamma_K$. Then we may argue as in the proof of lemma 2.2.20, to reduce the claim to the special case where $Z_{\mathfrak{q}} = Y$ and h is the identity map, therefore $U \subset Y$ is an open neighborhood of $y(s)$ and $t \in \mathscr{O}_Y(U)$. In such case, in view of claim 3.3.33(i), we may find $t' \in \kappa(a^{-1})$ such that $|t/t'|_{y(a^{-1})} = 1$. Then t/t' extends to a section of $\mathscr{O}^+_Y$ over an open subset $U' \subset U$ containing $y(a^{-1})$, and up to restricting U' we may assume that t/t' is invertible in $\mathscr{O}^+_Y(U')$. Applying again claim 3.3.33(i) we see that $|t|_{y(s)} = |t'|_{\eta(s)}$ for every $s \in [s, a^{-1}] \cap \Gamma_K$ such that $y(s) \in U'$, and then both assertions follow from lemma 2.2.20. $\Diamond$

Let $d := o(St_{\mathfrak{q}})$, and pick $t \in \kappa(z(a^{-1}))$ such that

$$|t|_{z(a^{-1})} = (1 - \varepsilon)^{1/d}.$$

By claim 3.3.34, there exists an open neighborhood $U \subset Z_{\mathfrak{q}}$ of $z(a^{-1})$ and $r \in (a, a^{-1})$ such that $t \in \mathscr{O}_Z(U)$, $z(s) \in U$ for every $s \in [r, a^{-1}] \cap \Gamma_K$ and :

$$|t|_{z(s)} = |t|_{z(a^{-1})} \cdot (s \cdot a)^{1/d} = (1 - \varepsilon)^{1/d} \cdot (s \cdot a)^{1/d} \qquad \text{for every } s \in [r, a^{-1}] \cap \Gamma_K. \tag{3.3.35}$$

Let $\sigma \in St(\mathfrak{q})$ be any element; by definition, we have :

$$i_{z(a^{-1})}(\sigma) = |t - \sigma(t)|_{z(a^{-1})}.$$

Now, let $s \in [r, a^{-1}] \cap \Gamma_K$, and choose $c \in K^\times$ such that $|c| = (s \cdot a)^{1/d}$; in view of (3.3.35), we also obtain the identity :

$$i_{z(s)}(\sigma) = |c^{-1} \cdot t - \sigma(c^{-1} \cdot t)|_{z(s)} = (s \cdot a)^{-1/d} \cdot |t - \sigma(t)|_{z(s)}. \tag{3.3.36}$$

Set $k := i_{z(a^{-1})}(\sigma)^\natural$; if we apply claim 3.3.34 to $t - \sigma(t)$, we can rewrite (3.3.36) in the form :

$$i_{z(s)}(\sigma) = (s \cdot a)^{k-1/d} \cdot |t - \sigma(t)|_{z(a^{-1})} = (s \cdot a)^{k-1/d} \cdot i_{z(a^{-1})}(\sigma)$$

which easily yields assertions (ii) and (iii). $\square$

3.3.37. Clearly, the analogue of theorem 3.3.29 holds also for the fibres over the points $\eta'(s)$ (see (3.3.11)). Namely, suppose that $r \in [a, b)$; then there exists $r' \in (r, a^{-1})$ and for every $s \in [r, r') \cap \Gamma_K$ a point $x'(s) \in f^{-1}(\eta'(s))$ such that :

- The stabilizer subgroup $St_{x'(s)} \subset G$ of $x'(s)$ under the natural G-action on $f^{-1}(\eta'(s))$, is independent of s.

- The length of the higher ramification filtration :

$$P_{\beta_m(s)} \subset \cdots \subset P_{\beta_1(s)} \subset St^{(p)}_{x'(s)}$$

of $St_{x'(s)} = \mathrm{Gal}(\kappa(x'(s))^{\wedge h}/\kappa'(s)^{\wedge h})$ is independent of $s \in [r, r') \cap \Gamma_K$.
- Set $\beta_k^\natural := \beta_k(r)^\natural$ for every $k \leq m$. Then :

$$\beta_k(s) = (s/r)^{-\beta_k^\natural} \cdot \beta_k(r) \qquad \text{for every } s \in [r, r') \cap \Gamma_K.$$

In other words, at the left (resp. at the right), of every $r \in (a, b) \cap \Gamma_K$, the higher ramification filtrations of the points lying over $\eta(s)$ (resp. $\eta'(s)$) for s sufficiently close to r, change in a continuous – indeed linear – fashion. To get the complete picture, we must also analyze what happens when we switch from the left to the right of a given radius r, *i.e.* we need to understand how the filtrations $(P_{\gamma_i(r)} \mid i = 1, \dots, n)$ and $(P_{\beta_i(r)} \mid i = 1, \dots, m)$ are related. The key is to compare both ramification filtrations to a third one, attached to the finite Galois extension $\kappa(r^\flat) \subset \kappa(x(r)^\flat) = \kappa(x'(r)^\flat)$ (see [30, Prop.1.5.4]). To this aim, we make the following :

Definition 3.3.38. Let $f : X \to \mathbb{D}(a, b)$ be a Galois finite étale covering, with Galois group G, $x \in X$ a point of type (III) and $x^\flat$ its unique proper generization (notation of (2.2.8)). Let $St_x^\flat \subset G$ be the stabilizer subgroup of the point $x^\flat$, under the natural action of G on $f^{-1}f(x^\flat)$. Then $St^\flat_x$ is naturally identified with the Galois group $\mathrm{Gal}(\kappa(x^\flat)^\wedge/\kappa(f(x^\flat))^\wedge)$. For any given $c \in K^+ \setminus \{0\}$ we set :

$$P^\flat_\gamma := \mathrm{Ker}(St^\flat_x \to \mathrm{Aut}(\kappa(x^\flat)^{\wedge +} \otimes_{K^+} K^+/cK^+)) \qquad \text{where } \gamma := |c|.$$

Clearly $P^\flat_\gamma$ is a normal subgroup of $St^\flat_x$ for every $\gamma \in \Gamma^+_K$. and the sequence $(P^\flat_\gamma \mid \gamma \in \Gamma^+_K)$ is called the *higher ramification filtration* of $St^\flat_x$.

Proposition 3.3.39. *Let $x \in X$ be as in* (3.3)*, and let $(P_\gamma \mid \gamma \in \Gamma^+_x)$ (resp. $(P^\flat_\gamma \mid \gamma \in \Gamma^+_K)$) be the higher ramification filtration of St_x (resp. of $St^\flat_x$). Then $St_x \subset St^\flat_x$ and moreover :*

$$P^\flat_\gamma = \bigcup_{n \in \mathbb{Z}} P_{\gamma_0^n \cdot \gamma} \qquad \textit{for every } \gamma \in \Gamma^+_K \setminus \{1\}.$$

(Here γ_0 is the largest element of $\Gamma^+_x \setminus \{1\}$.)

Proof. The first assertion is obvious. Next, let $\{x_1, \dots, x_k\} := f^{-1}f(x)$ be the orbit of $x = x_1$ under the G-action; in light of lemma 2.2.2, we see that $\kappa(x_i)^\wedge = \kappa(x^\flat)^\wedge$ for every $i \leq k$, and the rings $\kappa(x_i)^{\wedge +}$ are the valuation rings of the field $\kappa(x^\flat)^\wedge$ that dominate $\kappa(f(x))^{\wedge +}$. Let C be the integral closure of $\kappa(f(x))^{\wedge +}$ in $\kappa(x^\flat)^\wedge$; C is semilocal ring, whose localizations at the maximal ideals are the valuation rings $\kappa(x_i)^{\wedge +}$; applying [41, Th.1.4] we may then find $t \in C$ such that :

$$(3.3.40) \qquad |t|^\wedge_x = 1 \qquad \text{and} \qquad |t|^\wedge_{x_i} < 1 \quad \text{for every } i = 2, \dots, k.$$

Suppose now that $\sigma \in St^\flat_x \setminus St_x$; (3.3.40) implies easily that $|\sigma(t) - t|^\wedge_{x^\flat} = (|\sigma(t) - t|^\wedge_x)^\flat = 1$, hence $\sigma \notin P^\flat_\gamma$ whenever $\gamma < 1$, in other words :

$$(3.3.41) \qquad P^\flat_\gamma \subset St_x \qquad \text{for every } \gamma < 1.$$

Thus, suppose $\sigma \in P^\flat_\gamma$ for some $\gamma < 1$, and choose $t \in \kappa(x)^\wedge$ such that $|t|_x = \gamma_0$; due to (3.3.41), we have : $i_x(\sigma)^\flat = |\sigma(t) - t|^\flat_x \leq \gamma$, whence $P^\flat_\gamma \subset P'_\gamma := \bigcup_{n \in \mathbb{Z}} P_{\gamma_0^n \cdot \gamma}$.

Conversely, suppose $\sigma \in P_\gamma$ for some $\gamma \in \Gamma^+_x$ such that $\gamma^\flat < 1$; by definition, this means that $|\sigma(t) - t|_x < \gamma$ for every $t \in \kappa(x)^{\wedge +}$ ([31, Lemma 2.1(ii)]). Let $s \in \kappa(x^\flat)^{\wedge +}$; then $c \cdot s \in \kappa(x)^{\wedge +}$ for every $c \in \mathfrak{m}$, hence :

$$|c| \cdot |\sigma(s) - s| = |\sigma(c \cdot s) - c \cdot s|_x < \gamma$$

therefore $|\sigma(c\cdot s)-c\cdot s|_x<\gamma\cdot|c|^{-1}$ for every $c\in\mathfrak{m}\setminus\{0\}$ and consequently $|\sigma(s)-s|^\flat_x\leq\gamma^\flat$, *i.e.* $\sigma\in P_{\gamma^\flat}^\flat$, which shows that $P'_\gamma\subset P^\flat_\gamma$ for every $\gamma\in\Gamma^+_K\setminus\{1\}$, as stated. $\square$

Remark 3.3.42. If we apply proposition 3.3.39 to a pair of points $x,x'\in X$ lying over $\eta(r)$ and respectively $\eta'(r)$, and such that $x^\flat=x'^\flat$, we see that both ramification filtrations "to the left" and "to the right" of the radius r can be compared to the same "central" ramification filtration for $St^\flat_x=St^\flat_{x'}$. This expresses the sought continuity property for the jumps of the ramification filtrations.

4. Local systems on the punctured disc

In this chapter we fix a complete and algebraically closed valued field $(K,|\cdot|)$ of rank one and of zero characteristic. We shall use the standard notation of (2.2), and we suppose that the characteristic of the residue field $K^\sim$ is $p>0$.

4.1. **Break decomposition.** Let Λ be an artinian local $\mathbb{Z}[1/p]$-algebra whose residue field $\overline{\Lambda}$ has positive characteristic $\ell\neq p$; we assume that the group $\Lambda^\times$ of invertible elements of Λ contains a subgroup isomorphic to $\mu_{p^\infty}:=\bigcup_{n>0}\mu_{p^n}$ (where μ_{p^n} denotes the group of p^n-th roots of one contained in $K^\times$), and we fix such an imbedding :

$$\mu_{p^\infty}\subset\Lambda^\times. \tag{4.1.1}$$

Moreover, we shall also suppose that Λ is the filtered union of its finite subrings. This latter condition is motivated by the following :

Lemma 4.1.2. *Let X be a quasi-compact analytic adic space over K, F a locally constant constructible Λ-module on the étale site $X_{\text{ét}}$ of X. Then there exists a finite subring $\Lambda'\subset\Lambda$ and a locally constant constructible Λ'-module F' on $X_{\text{ét}}$ such that $F\simeq F'\otimes_\Lambda\Lambda'$.*

Proof. This lemma – stated in the language of Berkovich's non-archimedean analytic varieties – appears in [42, Lemma 4.1.8]. We sketch here the argument in the case of adic spaces. First, using [30, Lemma 1.4.7, Cor.1.7.4] we find finitely many affinoid open subsets $U_1,\dots,U_n\subset X$ covering X, and for each $i\leq n$ a finite étale morphism $f_i:V_i\to U_i$ such that $f_i^*F_{|U_i}$ is a constant Λ-module, whose stalk (at some chosen geometric point of V_i) we denote M_i. Then V_i is an affinoid adic space for every $i\leq n$ ([30, §1.4.4]), hence the same holds for $W_i:=V_i\times_{U_i}V_i$, especially the set $\pi_0(W_i)$ of connected components of W_i is finite. Then the descent datum for $F_{|U_i}$ relative to the morphism f_i amounts to a finite set of Λ-automorphisms $(\phi_{ij}\mid j\in\pi_0(W_i))$ of M_i, fulfilling a certain cocycle condition. Since M_i is of finite type and Λ is noetherian, we may find a finite subring $\Lambda_i\subset\Lambda$, a finite Λ_i-module M'_i and a set of automorphisms $(\phi'_{ij}\mid j\in\pi_0(W_i))$ such that $M_i\simeq M'_i\otimes_{\Lambda_i}\Lambda$ and $\phi_{ij}=\phi'_j\otimes_{\Lambda_i}\mathbf{1}_\Lambda$ for every $i\leq n$ and $j\in\pi_0(W_i)$. Furthermore, after replacing Λ_i by some larger finite subring, we can achieve that the cocycle conditions are still fulfilled by the system $(\phi'_{ij}\mid j\in\pi_0(W_i))$; hence the latter furnishes a descent datum for M'_i relative to f_i, whence a Λ_i-module F'_i on $U_{i,\text{ét}}$ such that $F'_i\otimes_{\Lambda_i}\Lambda\simeq F_{|U_i}$. Next, let $U_{ij}:=U_i\cap U_j$ for every $i,j\leq n$, so that F is defined by a cocycle system of isomorphisms $(F'_i\otimes_{\Lambda_i}\Lambda)_{|U_{ij}}\xrightarrow{\sim}(F'_j\otimes_{\Lambda_j}\Lambda)_{|U_{ij}}$. Again, these isomorphisms are already defined over some larger subring $\Lambda_{ij}\supset\Lambda_i+\Lambda_j$, and the claim follows easily. $\square$

Remark 4.1.3. (i) Keep the situation of lemma 4.1.2; an easy corollary is the following fact. There exists a finite étale covering $f:Y\to X$ such that f^*F is a constant Λ-module on $Y_{\text{ét}}$.

(ii) This is in general false, if Λ is not a filtered union of finite subrings : for a counterexample, consider an elliptic curve E over K, with bad reduction over K^+; it is well known that the associated analytic space E^{an} can be uniformized by the analytic torus $\mathbb{G}^{\text{an}}_{m,K}$, and the corresponding étale covering $\mathbb{G}^{\text{an}}_{m,K}\to E^{\text{an}}$ is Galois with group $G\simeq\mathbb{Z}$. If we take $\Lambda:=\mathbb{F}_\ell(T)$,

we may define an action $\chi : G \to \mathrm{End}_\Lambda(L)$ on a one-dimensional Λ-vector space L, by letting a generator $\sigma \in G$ act as multiplication by T. The character χ defines a locally constant constructible Λ-module on $E^{\mathrm{an}}_{\text{ét}}$ that trivializes on $\mathbb{G}^{\mathrm{an}}_{m,K}$, but does not trivialize on any finite covering of E^{an}.

4.1.4. Let F a constructible, locally constant and locally free sheaf of Λ-modules on the étale site of the *punctured disc* of radius one, *i.e.* the analytic adic space

$$\mathbb{D}(1)^* := \mathbb{D}(1) \setminus \{0\}$$

where $0 \in \mathbb{D}(1)$ is the closed point corresponding to the valuation given by the rule : $f(\xi) \mapsto |f(0)|$ for every $f \in A(1)$ (notation of (2.2.7)). Following [31, §8], for every $r \in \Gamma_K^+$ one defines the Swan conductor of the stalk $F_{\eta(r)}$, viewed as a representation of the Galois group of the algebraic closure of the henselian field $\kappa(r)^{\wedge h}$:

$$\pi_1(r) := \mathrm{Gal}((\kappa(r)^{\wedge h})^a : \kappa(r)^{\wedge h}). \tag{4.1.5}$$

This conductor is a (possibly negative) integer which we shall denote by :

$$\mathsf{sw}^\natural(F, r^+).$$

(Huber denotes this quantity by $\alpha_x(F)$ in *loc.cit.*, with $x := \eta(r)$.)

4.1.6. In this chapter we wish to investigate the properties of the function

$$-\log \Gamma_K^+ \to \mathbb{Z} \quad : \quad -\log r \mapsto \mathsf{sw}^\natural(F, r^+). \tag{4.1.7}$$

To start out, the identity ([31, Lemma 8.1(iii)]) :

$$\mathsf{sw}^\natural(F, r^+) = \mathrm{length}_\Lambda \Lambda \cdot \mathsf{sw}^\natural(F \otimes_\Lambda \overline{\Lambda}, r^+)$$

reduces the study of $\mathsf{sw}^\natural(F, r^+)$ to that of $\mathsf{sw}^\natural(F \otimes_\Lambda \overline{\Lambda}, r)$. Now, for every $r \in \Gamma_K^+$ let us fix a finite Galois étale covering $f_r : X_r \to \mathbb{D}(r, 1)$ such that $f_r^* F_{|\mathbb{D}(r,1)}$ is a constant sheaf (since $\mathbb{D}(r, 1)$ is a quasi-compact open subspace of $\mathbb{D}(1)^*$, the existence of f_r is ensured by remark 4.1.3(i)). Let G_r be the Galois group of the covering f_r; the sheaf $(F \otimes_\Lambda \overline{\Lambda})_{|\mathbb{D}(r,1)}$ can be regarded in the usual way as a finite dimensional $\overline{\Lambda}$-linear representation of G, in other words, as a positive element $\overline{\chi} \in K_0(\overline{\Lambda}[G])$. By inspecting the definitions we find the identity :

$$\mathsf{sw}^\natural(F, s^+) = \mathrm{length}_\Lambda \Lambda \cdot \langle \overline{\mathsf{sw}}^\natural_{G_r}(s^+), \overline{\chi} \rangle_{G_r} \qquad \text{for every } s \in (r, 1] \cap \Gamma_K.$$

This leads us to set :

$$\delta_F(-\log s) := \mathrm{length}_\Lambda \Lambda \cdot \delta_{f_r, \overline{\chi}}(-\log s) \qquad \text{for every } s \in (r, 1] \cap \Gamma_K$$

(notation of (3.3.25)). Suppose now that $f'_r : X'_r \to \mathbb{D}(r, 1)$ is another Galois covering that dominates f'_r (*i.e.* such that f'_r factors through f_r). Let G' be the Galois group of f'_r, and set $\overline{\chi}' := \mathrm{Res}^G_{G'} \overline{\chi}$; it follows easily from (3.3.6) that $\delta_{f_r, \overline{\chi}} = \delta_{f'_r, \overline{\chi}'}$. Since any two Galois étale coverings are dominated by a common one, we deduce that the function δ_F thus defined is independent of the choice of f. Especially, let $r' < r$ be another positive real number in Γ_K, choose a Galois étale covering $f_{r'} : X' \to \mathbb{D}(r', 1)$ trivializing F, let $\overline{\chi}'$ be the $\overline{\Lambda}[G_{r'}]$-module corresponding to $(F \otimes_\Lambda \overline{\Lambda})_{|\mathbb{D}(r',1)}$ and define the function $\delta'_F := \mathrm{length}_\Lambda \Lambda \cdot \delta_{f_{r'}, \overline{\chi}'} : (0, \log 1/r'] \cap \log \Gamma_K \to \mathbb{R}$; it follows easily that δ'_F agrees with δ_F wherever the latter is defined. Hence, by patching these locally defined function, we obtain a well defined mapping :

$$\delta_F : -\log \Gamma_K^+ \to \mathbb{R}.$$

Proposition 4.1.8. *In the situation of* (4.1.4)*, the mapping δ_F is the restriction of a non-negative, piecewise linear, continuous and convex real-valued function. Moreover :*

$$\frac{d\delta_F}{dt}(-\log r^+) = \mathsf{sw}^\natural(F, r^+) \qquad \textit{for every } r \in \Gamma_K^+.$$

Proof. All the hard work has already been done, and it remains only to invoke proposition 3.3.26. □

Corollary 4.1.9. *In the situation of* (4.1.4)*, the function :*

$$-\log\Gamma_K^+ \to \mathbb{Z} \qquad -\log r \mapsto \mathsf{sw}^\natural(F, r^+)$$

is monotonically non-decreasing, and moreover :

$$\mathsf{sw}^\natural(F, 0^+) := \lim_{r\to 0^+} \mathsf{sw}^\natural(F, r^+) \in \mathbb{N}\cup\{+\infty\}.$$

Proof. The monotonicity follows from the convexity of δ_F. The limit value $\mathsf{sw}^\natural(F, 0^+)$ cannot be negative, since δ_F is non-negative. □

4.1.10. To advance further, we use the break decomposition of [31, §8]. We choose the elegant presentation of N.Katz in [34, Ch.1], which makes it transparent that this is really a general representation-theoretic device. Indeed, suppose that H is a finite group with a unique (hence normal) p-Sylow subgroup P, and assume that P admits a finite descending filtration :

$$P_n := \{1\} \subset P_{n-1} \subset \cdots P_1 \subset P_0 := P$$

consisting of subgroups P_i normal in H for every $i \leq n$. Let R be any $\mathbb{Z}[1/p]$-algebra, and for every $i \leq n$ let us define the element :

$$e_i := \frac{1}{o(P_i)}\sum_{g\in P_i} g \in R[H].$$

Since P_i is normal in H, every e_i is a central idempotent element in $R[H]$. One verifies easily that the central idempotents :

$$e_0, \quad e_1\cdot(1-e_0), \quad e_2\cdot(1-e_1), \quad \cdots \quad , e_n\cdot(1-e_{n-1}) = 1-e_{n-1}$$

are orthogonal and sum to 1, hence define a natural decomposition of $R[H]$ in $n+1$ direct factors. Correspondingly, every $R[H]$-module M admits a *break decomposition* :

$$M = M_{-1}\oplus M_0\oplus\cdots\oplus M_{n-1}$$

into $R[H]$-submodules such that :

$$M_{-1} = M^P \qquad M_i^{P_i} = 0 \quad \text{for every } i \geq 0 \qquad \text{and} \qquad M_i^{P_j} = M_i \quad \text{whenever } j > i.$$

Furthermore, for every $i \leq n$ the rule $M \mapsto M_i$ defines an exact functor : $R[H]\text{-}\mathbf{Mod} \to R[H]\text{-}\mathbf{Mod}$, and for every pair of $R[H]$-modules M, N we have :

$$\mathrm{Hom}_{R[H]}(M_i, N_j) = 0 \qquad \text{whenever } i \neq j.$$

One deduces easily that :

$$\begin{array}{rl} M_i\otimes_R N_j \subset (M\otimes_R N)_{\max(i,j)} & \text{if } i\neq j \\ M_i\otimes_R N_i \subset \sum_{j\leq i}(M\otimes_R N)_j & \text{for every } i = -1,\dots,n \\ \mathrm{Hom}_R(M_i,N_j)\subset \mathrm{Hom}_R(M,N)_{\max(i,j)} & \text{if } i\neq j \\ \mathrm{Hom}_R(M_i,N_i)\subset \sum_{j\leq i}\mathrm{Hom}_R(M,N)_j & \text{for every } i=-1,\dots,n. \end{array} \tag{4.1.11}$$

See [34, Lemma 1.3] for details. Moreover, the break decomposition is invariant under arbitrary base-change $R \to R'$, *i.e.* we have :

$$(M\otimes_R R')_i = M_i\otimes_R R' \qquad \text{for every } i \leq n. \tag{4.1.12}$$

4.1.13. The generalities of (4.1.10) shall be applied to the group $H := St_x$ of (3.3) and its higher ramification filtration, and with $R := \Lambda$. However, for book-keeping purposes, it is convenient to replace the lower-numbering indexing of this filtration, by a upper-numbering one. With our current notation, this is defined as follows. First, one considers the order-preserving bijection :

$$\phi : \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_x \to \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_x \qquad \gamma \mapsto \prod_{g \in St_x} \max(\gamma, i(g)/\gamma_0)$$

where $\gamma_0 \in \Gamma_x^+$ is defined as in (2.2.14). Notice that ϕ maps $(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_x)^+$ bijectively onto itself. Next we let :

$$P^\gamma := P_{\phi^{-1}(\gamma)} \qquad \text{for every } \gamma \in \Gamma_x^+.$$

If $\gamma_1 > \cdots > \gamma_{n-1} > \gamma_n$ are the jumps in the family $(P_\gamma \mid \gamma \in \Gamma_x^+)$ that are less than 1, we obtain therefore a finite filtration of St_x :

$$\{1\} \subset P^{\phi(\gamma_n)} \subset P^{\phi(\gamma_{n-1})} \subset \cdots \subset P^{\phi(\gamma_1)} \subset P$$

where P is the p-Sylow subgroup of St_x. If now M is any Λ-module, we derive a break decomposition as in (4.1.10) :

$$M = M(1) \oplus M(\phi(\gamma_1)) \oplus \cdots \oplus M(\phi(\gamma_n))$$

such that $M(1) = M^P$ and :

(4.1.14)

$$M(\phi(\gamma_i))^{P^{\phi(\gamma_i)}} = 0 \quad \text{for every } i \leq n \quad \text{and} \quad M(\phi(\gamma_i))^{P^{\phi(\gamma_j)}} = M(\phi(\gamma_i)) \quad \text{whenever } j > i.$$

The values $\phi(\gamma_i)$ such that $M(\phi(\gamma_i)) \neq 0$ are called *the breaks* of M.

4.1.15. Especially, let F be as in (4.1.4), and $r \in \Gamma_K^+$. Choose $a \in (0, r) \cap \Gamma_K$ and a finite Galois étale covering $f_a : X_a \to \mathbb{D}(a, 1)$ trivializing $F_{|\mathbb{D}(a,1)}$. Denote by G_a the Galois group of f_a, pick any point $x \in f_a^{-1}(\eta(r))$, and let $St_x \subset G_a$ be the stabilizer of x. The stalk $F_r := F_{\eta(r)}$ is a $\Lambda[G_a]$-module of finite type, hence a $\Lambda[St_x]$-module, by restriction. Then the upper numbering filtration of St_x yields a break decomposition :

(4.1.16) $$F_r = F_r(\beta_0(r)) \oplus F_r(\beta_1(r)) \oplus \cdots \oplus F_r(\beta_n(r))$$

with $1 = \beta_0(r) > \beta_1(r) > \cdots > \beta_n(r)$. Of course, this is also a $\pi_1(r)$-equivariant decomposition (notation of (4.1.5)).

Lemma 4.1.17. *With the notation of* (4.1.15)*, we have the identity :*

$$\delta_F(-\log r) = -\sum_{i=1}^{n} \log \beta_i(r)^\flat \cdot \mathrm{length}_\Lambda(F_r(\beta_i(r))).$$

Proof. First, (4.1.12) allows to reduce to the case where $\Lambda = \overline{\Lambda}$. Then the sought identity is derived from (3.3.3) by a standard calculation (cp. the proof of [31, Prop.8.2 and Cor.8.4]). □

Lemma 4.1.18. *In the situation of* (4.1.15)*, choose $r' \in (a, r)$ such that the conditions* (i) *and* (ii) *of theorem* 3.3.29 *are fulfilled, with $f := f_a$. Then, for every $s \in (r', r] \cap \Gamma_K$:*

(i) *The length of the break decomposition of F_s :*

$$F_s = F_s(\beta_0(s)) \oplus F_s(\beta_1(s)) \oplus \cdots \oplus F_s(\beta_n(s))$$

is independent of s.

(ii) *For every other $s' \in (r', r] \cap \Gamma_K$, the isomorphism $\omega_{s,s'}$ of theorem* 3.3.29(i) *induces equivariant isomorphisms $F_s(\beta_k(s)) \simeq F_{s'}(\beta_k(s'))$, for every $k = 0, \ldots, n$.*

(iii) *Moreover, set $\beta_k^\natural := \beta_k(r)^\natural$ for every $k \leq n$. Then :*

$$\beta_k(s) = (s/r)^{\beta_k^\natural} \cdot \beta_k(r) \qquad \textit{for every } s \in (r', r] \cap \Gamma_K.$$

Proof. This is an exercise in translating from lower to upper numbering. Indeed, assertions (i) and (ii) are clear from theorem 3.3.29, and it remains only to show (iii). However, let $\{x(s) \mid s \in (r', r] \cap \Gamma_K\}$ be a family of points as in theorem 3.3.29 (with $f := f_a$), and $P_{\gamma_n(s)} \subset \cdots \subset P_{\gamma_1(s)} \subset St^{(p)}_{x(s)}$ the lower numbering ramification filtration of $St_{x(s)}$. Let also γ_0 be the largest element in $\Gamma^+_{x(s)} \setminus \{1\}$. Then, for every $k \leq n$ we may compute :

$$\begin{aligned}\beta_k(s) := \phi(\gamma_k(s)) &= \gamma_k(s)^{o(P_{\gamma_k(s)})} \cdot \prod_{g \in St_{x(s)} \setminus P_{\gamma_k(s)}} i(g)/\gamma_0 \\ &= \gamma_k(s)^{o(P_{\gamma_k(s)})} \cdot \prod_{1 \leq t < k} \gamma_t(s)^{o(P_{\gamma_t(s)}) - o(P_{\gamma_{t+1}(s)})}\end{aligned}$$

so the sought identities follow from theorem 3.3.29(iii). □

4.2. **Local systems with bounded ramification.** We keep the notation of (4.1). Corollary 4.1.9 suggests the following :

Definition 4.2.1. Let F be a locally constant and locally free Λ-module of finite rank on the étale site of $\mathbb{D}(1)^*$. We say that F has *bounded ramification* if $\mathrm{sw}^\natural(F, 0^+) \in \mathbb{N}$.

The class of sheaves with bounded ramification includes that of meromorphically ramified Λ-modules from [42]. The first result concerning these sheaves is :

Theorem 4.2.2. (i) *If F and F' are two Λ-modules with bounded ramification on $\mathbb{D}(1)^*_{\text{ét}}$, then $F \otimes_\Lambda F'$ and $\mathscr{H}om_\Lambda(F, F')$ have also bounded ramification.*

(ii) *Especially, if Λ is a field, the full subcategory of the category of Λ-modules on $\mathbb{D}(1)^*_{\text{ét}}$, consisting of all Λ-modules with bounded ramification, is tannakian.*

Proof. Of course (ii) follows from (i). To show assertion (i) for $F \otimes_\Lambda F'$, since we know *a priori* that $\delta_{F \otimes_\Lambda F'}$ is piecewise linear, continuous and convex (proposition 4.1.8), it suffices to provide a rough estimate on the rate of growth of the latter mapping, in terms of δ_F and $\delta_{F'}$. However, for given $r \in \Gamma^+_K$ let

$$F_r = F_r(1) \oplus F_r(\gamma_1) \oplus \cdots \oplus F_r(\gamma_n) \qquad \text{and} \qquad F'_r = F'_r(1) \oplus F'_r(\gamma'_1) \oplus \cdots \oplus F'_r(\gamma'_m)$$

be the break decompositions of the stalks of F and F' over the point $\eta(r)$. We set :

$$\lambda_i := \mathrm{length}_\Lambda F_r(\gamma_i) \qquad x_i := -\log \gamma_i^\flat \qquad \text{for every } i \leq n$$

and

$$\lambda'_j := \mathrm{length}_\Lambda F'_r(\gamma'_j) \qquad y_j := -\log \gamma'^\flat_j \qquad \text{for every } j \leq m.$$

Clearly $(F \otimes_\Lambda F')_{\eta(r)} = \oplus_{i,j} F_r(\gamma_i) \otimes_\Lambda F'_r(\gamma'_j)$, and using (4.1.11) and lemma 4.1.17 we arrive at the inequality :

$$\delta_{F \otimes_\Lambda F'}(-\log r) \leq \sum_{ij} x_i y_j \cdot \max(\lambda_i, \lambda'_j) \leq (\mathrm{rk}_\Lambda F) \cdot (\mathrm{rk}_\Lambda F') \cdot \max(\delta_F(-\log r), \delta_{F'}(-\log r))$$

as required. A similar argument takes care of $\mathscr{H}om_\Lambda(F, F')$ and concludes the proof of the theorem. □

Example 4.2.3. Choose a coordinate T on $\mathbb{A}^1_K$, (so that $\mathbb{A}^1_K = \mathrm{Spec}\, K[T]$), and for every $m \in \mathbb{N}$, denote by $f_m : \mathbb{A}^1_K \to \mathbb{A}^1_K$ the morphism such that $f^*_m(T) = T^m$. The restriction of f_m to $\mathbb{G}_{m,K} := \mathrm{Spec}\, K[T, T^{-1}]$ is a torsor in the étale topology of $\mathbb{G}_{m,K}$ for the group $\boldsymbol{\mu}_m \subset K^\times$, hence the analytification f^{ad}_m is a $\boldsymbol{\mu}_m$-torsor in the étale topology of $\mathbb{G}^{\mathrm{ad}}_{m,K}$. For every character $\chi : \boldsymbol{\mu}_m \to \Lambda^\times$, we let $\mathscr{K}(\chi)$ be the locally free rank one Λ-module on $\mathbb{D}(1)^*_{\text{ét}}$ which is induced, via χ, by the restriction of the torsor f^{ad}_m. Let $n \leq m$ be the order of χ (*i.e.* the smallest $k \in \mathbb{N}$

such that χ^k is the trivial character), and denote by μ a chosen generator of $\boldsymbol{\mu}_n$. It follows from [31, Ex.8.8] that :

$$\mathsf{sw}^\natural(\mathscr{K}(\chi), r^+) = 0 \qquad \text{for every } r \in \Gamma_K^+.$$

Especially, $\mathscr{K}(\chi)$ is a Λ-module with bounded ramification. Moreover, if χ is not trivial, the (unique) break of $\mathscr{K}(\chi)$ equals $|1-\mu|$ for every $r \in \Gamma_K^+$.

Example 4.2.4. Keep the notation of example 4.2.3, and let :

$$\mathbb{D}(1^-) := \bigcup_{r \in \Gamma_K^+ \setminus \{1\}} \mathbb{D}(r).$$

The morphism :

$$\log : \mathbb{D}(1^-) \to (\mathbb{A}^1_K)^{\mathrm{ad}} \quad \text{such that} \quad \log^*(T) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot T^n$$

is a torsor in the étale topology of $(\mathbb{A}^1_K)^{\mathrm{ad}}$ for the constant group μ_{p^∞} (see [31, Lemma 9.4] or [42, Lemma 6.1.1]). We denote by $\mathscr{L}$ the locally free Λ-module on $(\mathbb{A}^1_K)^{\mathrm{ad}}_{\text{ét}}$ which is induced by this torsor via the inclusion (4.1.1). The sheaf $\mathscr{L}$ has been studied at length in [42]. For instance, one can show that for any two morphisms $\phi, \psi : (\mathbb{A}^1_K)^{\mathrm{ad}} \to (\mathbb{A}^1_K)^{\mathrm{ad}}$, there exists a natural isomorphism :

$$(\phi^*\mathscr{L}) \otimes_\Lambda (\psi^*\mathscr{L}) \simeq (\phi+\psi)^*\mathscr{L} \tag{4.2.5}$$

where $\phi+\psi$ is the addition of ϕ and ψ, regarded as sections of the structure sheaf of $(\mathbb{A}^1_K)^{\mathrm{ad}}$.

Let $g : \mathbb{G}_{m,K} \to \mathbb{A}^1_K$ be the morphism such that $g^*(T) = T^{-1}$, and for every $q \in \mathbb{Q}_{\geq 0}$, let $(m,n) \in \mathbb{N}^2$ be the unique pair of relatively prime integers such that $q = n/m$; we set :

$$\mathscr{L}(q) := f_{m*} \circ f_n^* \circ g^*\mathscr{L}$$

where f_m and f_n are defined as in example 4.2.3. The following lemma 4.2.6 shows that the sheaves $\mathscr{L}(q)_{|\mathbb{D}(1)^*}$ have bounded ramification.

Lemma 4.2.6. *Let $q \in \mathbb{Q}_{>0}$, and write $q = n/m$, with $n, m \in \mathbb{N}$ and $(n,m) = 1$; moreover, write $n = p^a N$, $m = p^b M$, with $a, b \geq 0$ and $(N,p) = (M,p) = 1$ (of course, either $a = 0$ or $b = 0$). Set $\lambda := |p|^{1/(p-1)}$ and $l := \mathrm{length}_\Lambda \Lambda$. The following holds (notice that $\delta_{\mathscr{L}(q)}(\rho)$ is defined for every $\rho \in \log \Gamma_K$) :*

- (i) *$\delta_{\mathscr{L}(q)}$ is the unique continuous piecewise linear function such that :*
  - (a) *$\delta_{\mathscr{L}(q)}(\rho) = 0$ whenever $\rho \leq q^{-1} \log \lambda$.*
  - (b) *The right slope of $\delta_{\mathscr{L}(q)}$ equals :*
    - *lN on the interval $[q^{-1}\log\lambda, q^{-1}\log|1/p||) \cap \log\Gamma_K$;*
    - *lp^jN on the interval $[jq^{-1}\log|1/p|, (j+1)q^{-1}\log|1/p|) \cap \log\Gamma_K$, for every $j = 1, \dots, a-1$.*
    - *ln on the half-line $[aq^{-1}\log|1/p|, +\infty) \cap \log\Gamma_K$.*
- (ii) $\mathsf{sw}^\natural(\mathscr{L}(q), 0^+) = ln$.
- (iii) *For every $q \in \mathbb{Q}_{\geq 0}$, the sheaf $\mathscr{L}(q)_{|\mathbb{D}(1)^*}$ is indecomposable in the category of locally free Λ-modules on $\mathbb{D}(1)^*_{\text{ét}}$.*
- (iv) *More precisely, for every $r \in \Gamma_K$, let $\pi_1(r)^{(p)}$ be the unique p-Sylow subgroup of $\pi_1(r)$. Set :*

$$r_0 := \begin{cases} |p|^{(b-1)/q} & \text{if } b \neq 0 \\ \lambda^{-1/q} & \text{if } b = 0. \end{cases}$$

Then, for every $r \leq r_0$, the stalk $\mathscr{L}(q)_r$ is an indecomposable $\Lambda[\pi_1(r)^{(p)}]$-module (notation of (4.1.15)); especially, $\mathscr{L}(q)_r$ has a unique break $\beta(q,r)$.

(v) *Suppose that* $a = b = 0$. *Then :*

$$\beta(q,r) = \begin{cases} r^q \cdot (1-\varepsilon)^q \cdot \lambda \cdot |p|^{-a} & \textit{for } r \leq \lambda^{-1/q} \\ 1 & \textit{otherwise}. \end{cases}$$

Proof. Without loss of generality, we may assume that Λ is a field, hence $l = 1$. Notice that $\mathscr{L}$ is trivial on the disc $\mathbb{D}(\lambda^-) := \bigcup_{r<\lambda} \mathbb{D}(r)$, hence :

$$\delta_{\mathscr{L}(1)}(\rho) = 0 \qquad \text{whenever } \rho < \log \lambda.$$

In view of proposition 4.1.8, $\delta_{\mathscr{L}(1)}$ is then completely determined, once we know its right derivative $\mathsf{sw}^\natural(\mathscr{L}(1), \cdot)$. However, [31, Lemma 9.4] shows that :

$$\mathsf{sw}^\natural(\mathscr{L}(1), r^+) = 1 \qquad \text{whenever } r \leq \lambda^{-1}$$

which gives (i), for $q = 1$. Suppose now that $n = p^a N > 1$ is an integer, and set $P := p^a$; according to [31, Ex.8.8(i)], we have :

$$\mathsf{sw}^\natural(\mathscr{L}(n), r^+) = \mathsf{sw}^\natural(f_{k*}\mathscr{L}(n), r^{n+}) \qquad \text{for every } r \in \Gamma_K \text{ and every } k \in \mathbb{N}$$

which translates as the identity :

$$\frac{d\delta_{\mathscr{L}(n)}}{dt}(\rho^+) = \frac{d\delta_{f_{k*}\mathscr{L}(n)}}{dt}(k\rho^+) \qquad \text{for every } \rho \in \log \Gamma_K \text{ and every } k \in \mathbb{N}. \tag{4.2.7}$$

We shall apply (4.2.7) with $k = n$, so we need to calculate the conductor of $f_{n*}\mathscr{L}(n)$. The projection formula yields :

$$f_{n*}\mathscr{L}(n) \simeq \mathscr{L}(1) \otimes_\Lambda f_{n*}\Lambda.$$

Claim 4.2.8. There is a natural isomorphism :

$$f_{n*}\Lambda \simeq f_{N*}\Lambda \otimes_\Lambda f_{P*}\Lambda.$$

Proof of the claim. The morphism f_n induces an inclusion of fundamental groups $\phi : H := \pi_1(\mathbb{G}_{m,K}, x) \to G := \pi_1(\mathbb{G}_{m,K}, f_n(x))$ (for any choice of a geometric point x) whose image is a normal subgroup with cokernel isomorphic to $\mathbb{Z}/n\mathbb{Z}$. The constant sheaf Λ on $\mathbb{G}_{m,K}$ corresponds to the trivial representation of H, and $f_{n*}\Lambda$ is the induction of this representation along the inclusion ϕ. Likewise we may describe $f_{N*}\Lambda$ and $f_{P*}\Lambda$. However :

$$\Lambda \otimes_{\Lambda[H]} \Lambda[G] \simeq \Lambda[G/H] = \Lambda[\mathbb{Z}/n\mathbb{Z}]$$

so $f_{n*}\Lambda$ is also the induction of the trivial representation of the trivial group $\{0\}$, along the inclusion $\{0\} \subset \mathbb{Z}/n\mathbb{Z}$, and likewise for $f_{N*}\Lambda$ and $f_{P*}\Lambda$. So finally, the sought isomorphism boils down to the Λ-algebra isomorphism : $\Lambda[\mathbb{Z}/n\mathbb{Z}] \simeq \Lambda[\mathbb{Z}/P\mathbb{Z}] \otimes_\Lambda \Lambda[\mathbb{Z}/N\mathbb{Z}]$. $\diamondsuit$

Since $\mu_{p^\infty} \subset \Lambda^\times$, we have :

$$f_{P*}\Lambda \simeq \bigoplus_{\chi:\boldsymbol{\mu}_P \to \mu_{p^\infty}} \mathscr{K}(\chi)$$

where the sum runs over the P different characters of $\boldsymbol{\mu}_P \subset K^\times$ (notation of example 4.2.3). Thus, $f_{n*}\mathscr{L}(n)$ decomposes as the direct sum of P terms of the form

$$\mathscr{M}(\chi) := \mathscr{L}(1) \otimes_\Lambda \mathscr{K}(\chi) \otimes_\Lambda f_{M*}\Lambda.$$

Let $p^j > 1$ be the order of the character χ; according to [31, Ex.8.8(ii)], the unique break of $\mathscr{K}(\chi)_r$ is independent of r, and equals $|p|^j \cdot \lambda$. On the other hand, the unique break $\beta(r)$ of $\mathscr{L}(1)_r$ can be computed from $\delta_{\mathscr{L}(1)}$ using proposition 4.1.8 : we get $\beta(r) = 1$ for $r > \lambda^{-1}$ and $\beta(r) = r\lambda \cdot (1-\varepsilon)$ for $r \leq \lambda^{-1}$. From (4.1.11) we deduce that the unique break of $(\mathscr{L}(1) \otimes_\Lambda \mathscr{K}(\chi))_r$ equals 1 for $r > |p|^j$ and $r\lambda \cdot (1-\varepsilon)$ for $r \leq |p|^j$. Next, since $(N,p) = 1$,

the only possible break of the stalk $(f_{N*}\Lambda)_r$ equals 1, hence the stalks of $\mathscr{L}(1)\otimes_\Lambda \mathscr{K}(\chi)$ and $\mathscr{M}(\chi)$ have the same breaks. Consequently :

$$\frac{d\delta_{\mathscr{M}(\chi)}(\rho^+)}{dt} = \begin{cases} 0 & \text{for } \rho < j\log|1/p| \\ N & \text{otherwise} \end{cases} \tag{4.2.9}$$

Since $\delta_{f_{n*}\mathscr{L}(n)} = \sum_\chi \delta_{\mathscr{M}(\chi)}$, assertion (i) for $q = n$ follows from (4.2.7) (with $k := n$) and (4.2.9). Finally, let $q := n/m$, with n, m two relatively prime integers; in order to determine the right slope of $\delta_{\mathscr{L}(q)}$, it suffices to apply (4.2.7) with $k := m$. This completes the proof of (i).

Assertion (ii) is an immediate consequence of (i); also (iii) follows directly from (iv), and (v) follows from (i) and (iv). Hence it remains only to show (iv) when Λ is a field, which we may assume to be algebraically closed. The assertion is obvious if q is an integer, since in that case $\mathscr{L}(q)_r$ has rank one. For the general case $q = n/m$, notice that the action of $\pi_1(s)$ (resp. $\pi_1(s^m)$) on $\mathscr{L}(n)_s$ (resp. $\mathscr{L}(q)_{s^m}$) factors through a finite quotient H_s (resp. G_s), and :

$$\mathscr{L}(q)_{s^m} \simeq \mathrm{Ind}_{H_s}^{G_s}\mathscr{L}(n)_s \qquad \text{for every } s \in \Gamma_K.$$

The morphism f_m is a torsor for the group $\boldsymbol{\mu}_m$, and we have a natural identification $G_s/H_s \simeq \boldsymbol{\mu}_m$. We shall apply Mackey's irreducibility criterion (this is shown in [47, §7.4, Cor.] in case the base field has characteristic zero, but the result holds whenever the characteristic of the algebraically closed field Λ does not divide the order of G_s; this latter condition is clearly fulfilled here). To this aim, we have to show that, for every $\mu \in \boldsymbol{\mu}_m \setminus \{1\}$, the conjugate representation $\mathscr{L}(n)_s^\mu$ is not isomorphic to $\mathscr{L}(n)_s$. However, we have a natural identification :

$$\mathscr{L}(n)_s^\mu \simeq \mu^*\mathscr{L}(n)_s$$

where $\mu : \mathbb{G}_{m,K} := \mathrm{Spec} K[T, T^{-1}] \to \mathbb{G}_{m,K}$ is the morphism such that $\mu^*(T) = \mu\cdot T$. According to (4.2.5), we have a natural isomorphism :

$$\mu^*\mathscr{L}(n)\otimes_\Lambda \mathscr{L}(n)^{-1} \simeq g^*\mathscr{L}$$

where $g : \mathbb{G}_{m,K} \to \mathbb{G}_{m,K}$ is the morphism such that $g^*(T) = (1-\mu^{-n})T^{-n}$. Since $\mathscr{L}_r$ is not trivial whenever $r \geq \lambda$, it follows that $(g^*\mathscr{L})_s$ is not trivial whenever $|1-\mu^{-n}|\cdot s^{-n} \geq \lambda$. However, μ is a primitive p^b-root of unity for some $j = 1, \dots, b$, we have :

$$|1-\mu| = \lambda\cdot|p|^{j-1}$$

so in this case, $(g^*\mathscr{L})_s$ is not trivial for $s \leq |p|^{(j-1)/n}$. If $b = 0$, then $|1-\mu| = 1$, and then $(g^*\mathscr{L})_s$ is not trivial for $s \leq \lambda^{-1/n}$. Letting $r := s^m$, we obtain the contention, in either case. As an immediate consequence, we see that $\mathscr{L}(q)_r$ admits a single break $\beta(q,r)$ for $r \leq r_0$; this break can be determined by evaluating $\delta_{\mathscr{L}(q)}(-\log r)$, since the latter must equal $-m\log\beta(q,r)^\flat$ (lemma 4.1.17); we leave to the reader the elementary calculation. □

4.2.10. Let F be a locally free Λ-module on $\mathbb{D}(1)^*$ with bounded ramification. We wish to define the *breaks of F around the origin*. Ideally, one would like to define a stalk $F_{\eta(0)}$ that captures the behaviour of F in arbitrarily small punctured open discs $\mathbb{D}(\varepsilon)^*$ centered at the origin; then the sought breaks should be numerical invariants associated to this stalk. To make sense of this, one would like to complete somehow the sequence of points $(\eta(r) \mid r \in \Gamma_K^+)$ with a limit point $\eta(0)$; however, such a limit point seems to elude the grasp of the formalism of adic spaces, hence we have to proceed in a rather more indirect fashion. But the ideal picture should be kept in mind, as it motivates much of what we are trying to do in the remainder of this work.

To begin with, for given $r, \alpha \in \Gamma_K^+$ we set :

$$F_r^\flat(\alpha) := \bigoplus_{\beta_i(r)^\flat = \alpha} F_r(\beta_i(r)) \tag{4.2.11}$$

where $1 = \beta_0(r) > \cdots > \beta_n(r)$ are the breaks of F_r, so that we have the break decomposition (4.1.16). Say that $\mathrm{rk}_\Lambda F = d$ and $-\log r = \rho$; we consider the unique sequence of real numbers:

$$0 \leq f_1(\rho) \leq f_2(\rho) \leq \cdots \leq f_d(\rho) \tag{4.2.12}$$

in which, for every $\beta \in \Gamma_K^+$, the value $-\log \beta$ appears with multiplicity equal to $\mathrm{rk}_\Lambda F_r^\flat(\beta)$.

Lemma 4.2.13. *The functions $f_1, \dots, f_d$ extend to piecewise linear continuous maps $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.*

Proof. Using lemma 4.1.18, we deduce already that $f_1, \dots, f_d$ extend to continuous, linear functions on every small segment of the form $[-\log r, -\log r')$.

It remains to show that for every $\rho > 0$ there is some small segment $(\rho', \rho]$ on which the functions f_i are continuous. To this aim, we remark that all the considerations of (4.2.10) and (4.1.13) can be repeated for the family of stalks $F_{\eta'(r)}$ (instead of $F_r := F_{\eta(r)}$). We obtain in this way a break decomposition $F_{\eta'(r)}(\delta_0) \oplus \cdots \oplus F_{\eta'(r)}(\delta_l)$ for $F_{\eta'(r)}$, and we may define the submodules $F^\flat_{\eta'(r)}(\alpha)$ for every $\alpha \in \Gamma_K^+$, just as in (4.2.11). Using the ranks of the modules $F^\flat_{\eta'(r)}(\alpha)$, we may finally construct a non-decreasing sequence $0 \leq f'_1(\rho) \leq f'_2(\rho) \leq \cdots \leq f'_d(\rho)$ analogous to (4.2.12). Making use of (3.3.37) (rather than theorem 3.3.29), one can then show the analogue of lemma 4.1.18 which expresses the continuity of the breaks δ_i; from the latter, we see that the functions f'_i are continuous on segments of the form $(\rho', \rho]$. To conclude it suffices to show that $f_i = f'_i$ for every $i \leq d$. This boils down to the following :

Claim 4.2.14. $F^\flat_{\eta'(r)}(\alpha) \simeq F^\flat_r(\alpha)$ for every $\alpha \leq 1$.

Proof of the claim. We do not only assert the existence of an isomorphism in the category Λ-$\mathbf{Mod}$, but more precisely, that the two modules are equivariantly isomorphic, in the following sense. Say that $r \in (a, b)$ for some $a, b \in \Gamma_K^+$, and pick a Galois étale covering $f : X \to \mathbb{D}(a, b)$ that trivializes $F_{|\mathbb{D}(a,b)}$; choose also points x, x' lying over repectively $\eta(r)$ and $\eta'(r)$, such that $x^\flat = x'^\flat$. We shall use the ramification filtration $(P^\flat_\gamma \mid \gamma \in \Gamma_K^+)$ of $St^\flat_x$, given by definition 3.3.38. According to proposition 3.3.39, the p-Sylow subgroup $St_x^{\flat(p)}$ is naturally a subgroup of $St_x^{(p)}$ and $St_{x'}^{(p)}$, and the claim amounts to a $St_x^{\flat(p)}$-equivariant identification of $F^\flat_{\eta'(r)}(\alpha)$ and $F^\flat_r(\alpha)$.

Proceeding as in (4.1.13), we replace the lower-numbering indexing by the upper-numbering $(P^{\flat,\gamma} \mid \gamma \in \Gamma_K^+)$. Say that $(P^\gamma \mid \gamma \in \Gamma_x^+)$ (resp. $(Q^\gamma \mid \gamma \in \Gamma_{x'}^+)$) is the upper-numbering ramification filtration for St_x (resp. for $St_{x'}$). Then proposition 3.3.39 yields the identities :

$$\bigcup_{n \in \mathbb{Z}} Q^{\gamma_0^n \cdot \gamma} = P^{\flat,\gamma} = \bigcup_{n \in \mathbb{Z}} P^{\gamma_0^n \cdot \gamma} \qquad \text{for every } \gamma \in \Gamma_K^+ \setminus \{1\}. \tag{4.2.15}$$

Recall that, by construction, for each $\beta \in \{\beta_1(r), \dots, \beta_n(r)\}$, the direct summand $F_r(\beta)$ is of the form $e_\beta \cdot F_r$, where e_β is a certain central idempotent in $\Lambda[St_x^{(p)}]$. Likewise, for every $\delta \in \{\delta_1, \dots, \delta_l\}$, we have $F_{\eta'(r)}(\delta) = e'_\delta \cdot F_{\eta'(r)}$ for a certain central idempotent $e'_\delta \in \Lambda[St_{x'}^{(p)}]$. Then, by inspecting the definitions, we see that e_β (resp. e'_δ) actually lies in the subring $\Lambda[St_x^{\flat(p)}]$, whenever $\beta^\flat < 1$ (resp. $\delta^\flat < 1$). (Here $St_x^{\flat(p)}$ is the unique p-Sylow subgroup of $t_x^\flat$.) Moreover, (4.2.15) leads to the identities :

$$\sum_{\beta^\flat = \alpha} e_\beta = \sum_{\delta^\flat = \alpha} e'_\delta \qquad \text{for every } \beta \in \Gamma_K^+ \setminus \{1\}.$$

This already shows the claim for $\alpha < 1$. The case for $\alpha = 1$ can similarly be dealt with, using (4.2.15) and the characterization (4.1.14) of the breaks : the details shall be left to the reader. $\square$

4.2.16. We shall denote by $\Delta(F) \subset \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ the union of the graphs of the functions $f_1, \dots, f_d$ defined in (4.2.12). Let now $(K', |\cdot|')$ be an algebraically closed valued field extension of $(K, |\cdot|)$ whose value group is $\mathbb{R}_{>0}$ (*e.g.* we can take a maximally complete field containing K). The given Λ-module F pulls back to a locally constant Λ-module F' on the adic space $\mathbb{D}(1)^* \times_{\operatorname{Spa} K} \operatorname{Spa} K'$. In view of lemma 3.3.8, we see that for every $r \in \Gamma_K^+$ the breaks of F'_r are the same as that of F_r, therefore the subset $\Delta(F')$ is none else than the topological closure of $\Delta(F)$. Hence for the considerations that follow we may replace K by K' and F by F', and assume that $\Gamma_K = \mathbb{R}_{>0}$. Simple operations on F can be translated into corresponding changes for the subset $\Delta(F)$. For instance, for any $s \in K^+ \setminus \{0\}$, let $\mu_s : \mathbb{D}(1)^* \to \mathbb{D}(1)^*$ be the "shrinking" morphism such $\mu_s^*(\xi) = s \cdot \xi$. We have $(\mu_s^* F)_x \simeq F_{\mu_s(x)}$ for every $x \in \mathbb{D}(1)^*$, so that

$$\Delta(\mu_s^* F) = (\log|s|, 0) + \Delta(F) := \mathbb{R}^2_{\geq 0} \cap \{(x + \log|s|, y) \mid (x, y) \in \Delta(F)\}.$$

We are now ready to make the following :

Definition 4.2.17. Assume that $\Gamma_K = \mathbb{R}_{>0}$ and let F be a locally constant and locally free Λ-module on $\mathbb{D}(1)^*_{\text{ét}}$ of finite rank. Then the *break function* of F is the mapping :

$$\beta(F, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$$

defined by the rule :

$$\beta(F, q) := \frac{1}{\mathrm{rk}_\Lambda \mathscr{L}(q)} \cdot \sup\{\mathsf{sw}^\natural(F \otimes_\Lambda \mu_s^* \mathscr{L}(q), 0^+) \mid s \in K^+ \setminus \{0\}\} \qquad \text{for every } q \in \mathbb{Q}_{\geq 0}.$$

Remark 4.2.18. Suppose that F is the restriction of a sheaf F' of Λ-modules on $(\mathbb{A}^1_K)^{\text{ad}}_{\text{ét}}$, and consider the case $q = 1$: the cohomology complex $R\Gamma_c(F' \mu_s^* \mathscr{L}(1))$ is none else than the stalk over the point $s^{-1} \in K = \mathbb{A}^1_K(K)$ of the Fourier transform $\mathscr{F}(F')$ of F, as defined in [42]. In view of the (analogue of the) Grothendieck-Ogg-Shafarevich formula (see [31, Th.10.2]), we see that the function $\beta(F, 1)$ essentially calculates the Euler-Poincaré characteristic of $\mathscr{F}(F)$ in a neighborhood of the point $\infty \in (\mathbb{P}^1_K)^{\text{ad}}$. This is the sort of quantities that appear in the method of stationary phase (see the introduction, §0.9), and indeed this sort of sheaf-theoretic harmonic analysis has motivated the definition of the function β.

For any rational number q, we define the *denominator* of q as the smallest positive integer n such that $nq \in \mathbb{Z}$.

Theorem 4.2.19. *Let F be as in definition* 4.2.17*, and suppose moreover that F has bounded ramification. Then :*

(i) *For every $q \in \mathbb{Q}_{\geq 0}$, $\beta(F, q)$ is a positive rational number, whose denominator divides the denominator of q.*
(ii) *$\beta(F, q) \geq q \cdot \mathrm{rk}_\Lambda F$ for every $q \in \mathbb{Q}_{\geq 0}$, and the inequality is an equality for every sufficiently large q (so $\beta(F, \cdot)$ is eventually linear).*
(iii) *The break function $\beta(F, \cdot)$ is the restriction of a function $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which is convex, continuous, non-decreasing and piecewise linear whose slopes are integers.*

Proof. Without loss of generality, we may assume that Λ is a field. We begin by introducing some notation : we let $\mathbb{S} \subset \mathbb{Q}$ be the subset of the numbers of the form n/m where n, m are relatively prime positive integers, such that $(p, n) = (p, m) = 1$. Also, for every $s \in K^\times$ and $q \in \mathbb{R}$, set

$$c(q, s) := (p-1)^{-1} \log|p| + q \log|s|.$$

By lemma 4.2.6(v), for any $q \in \mathbb{S}$, and every $s \in K^\times$, the subset $\Delta(\mu_s^* \mathscr{L}(q))$ is the graph of the function :

$$\rho \mapsto b(q, \rho, s) := \max\{0, q\rho - c(q, s)\} \qquad \text{for every } \rho \in \mathbb{R}_{\geq 0}.$$

In the study of the "stalk over $\eta(0)$", we are allowed to disregard the behaviour of our sheaf F outside any given punctured disc $\mathbb{D}(\varepsilon)^*$, *i.e.* we may disregard the part of $\Delta(F)$ that lies in a vertical band of the form $[0,c]\times\mathbb{R}$; hence, let us define $\Sigma(q)$ as the subset of all $c(q,s)\in\mathbb{R}$ such that

$$\Delta(\mu_s^*\mathscr{L}(q))\cap\Delta(F)\cap([q^{-1}c(q,s),+\infty)\times\mathbb{R})$$

is a set whose cardinality is at most countable. Notice that $\mathbb{R}\setminus\Sigma(q)$ has at most countable cardinality; especially, $\Sigma(q)$ is dense in $\mathbb{R}$, for every $q\in\mathbb{S}$.

Let $s\in K^+\setminus\{0\}$, $q\in\mathbb{S}$, set $\mathscr{L}':=\mu_s^*\mathscr{L}(q)$, and suppose that $c(q,s)\in\Sigma(q)$; this means that for every $\rho\geq\max(q^{-1}c(q,s),0)$:

- either $(\rho,b(q,\rho,s))\notin\Delta(F)$,
- or else the right and left slope of $b(q,\cdot,s)$ at the point ρ are different from the slopes of each of the functions f_i as in (4.2.12), such that $f_i(\rho)=b(q,\rho,s)$.

However, say that $\rho=-\log r$, let γ be the unique break of $\mathscr{L}'_r$, and $\beta_1,\dots,\beta_k$ the finitely many breaks of F_r; then by definition, $\Delta(F)\cap(\{r\}\times\mathbb{R})$ consists of the values $-\log\beta_j^\flat$ (for $j=1,\dots,k$), and $-\log\gamma^\flat=b(q,\rho,s)$. Furthermore, by theorem 3.3.29 and (3.3.37), the (right and left) slopes of the functions f_i at the point ρ are none else than the values $\beta_j^\natural$ (and likewise for the slope of $b(q,\cdot,s)$). We conclude that $\gamma\notin\{\beta_1,\dots\beta_k\}$, and then (4.1.11) implies that the breaks of $(F\otimes_\Lambda\mathscr{L}')_r$ are the values

$$\beta'_j:=\min(\gamma,\beta_j)\qquad\text{for } j=1,\dots,k. \tag{4.2.20}$$

Moreover, let

$$M(\beta_1)\oplus\cdots\oplus M(\beta_k)\qquad M'(\beta'_1)\oplus\cdots\oplus M'(\beta'_k)$$

be the break decompositions of F_r and respectively $(F\otimes_\Lambda\mathscr{L}')_r$; then :

$$\mathrm{rk}_\Lambda M'(\beta'_j)=\mathrm{rk}_\Lambda\mathscr{L}(q)\cdot\mathrm{rk}_\Lambda M(\beta_j)\qquad\text{for every } j\leq k. \tag{4.2.21}$$

Set $d:=\mathrm{rk}_\Lambda F$; combining lemma 4.2.6(iii), (4.2.20) and (4.2.21) we arrive at the identity :

$$\delta_{F\otimes_\Lambda\mathscr{L}'}(\rho)=\mathrm{rk}_\Lambda\mathscr{L}(q)\cdot\sum_{i=1}^d\max(f_i(\rho),b(q,\rho,s)).$$

Notice that, since $f_i\geq 0$ for every $i\leq d$, the foregoing identity persists also for $\rho<q^{-1}c(q,s)$. Recall also that these functions f_i are continuous and piecewise linear (lemma 4.2.13). This motivates the following :

Claim 4.2.22. For every $q\in\mathbb{R}_{\geq0}$ and $c\in\mathbb{R}$, consider the function

$$f_{q,c}:\mathbb{R}_{\geq0}\to\mathbb{R}_{\geq0}\qquad\rho\mapsto\sum_{i=1}^d\max(f_i(\rho),q\rho-c).$$

Then :

(i) $f_{q,c}$ is continuous, convex and piecewise linear.
(ii) The (right and left) slopes of $f_{q,c}$ are of the form $qa+b$, where $a\in\{0,1,\dots,d\}$, $b\in\mathbb{Z}$.
(iii) Moreover, $f_{q,c}$ is eventually linear (*i.e.* of the form $\rho\mapsto\rho x+y$ for every sufficiently large ρ).
(iv) More precisely, if $q\in\mathbb{Q}_{\geq0}$, then for every sufficiently large $\rho\in\mathbb{R}_{\geq0}$ the left and right slope of $f_{q,c}$ coincide, and their common value is a rational number whose denominator divides the denominator of q.
(v) For every $\rho\in\mathbb{R}_{\geq0}$, the function $\mathbb{R}_{\geq0}\to\mathbb{R}_{\geq0}:q\mapsto f_{q,c}(\rho)$ is non-decreasing, convex, continuous and piecewise linear.

Proof of the claim. By construction, the function $\mathrm{rk}_\Lambda \mathscr{L}(q) \cdot f_{q,c}$ is the mapping $\delta_{F\otimes_\Lambda \mathscr{L}'}$, whenever $q \in \mathbb{S}$, $\mathscr{L}' = \mu_s^* \mathscr{L}(q)$ and $c = c(q,s) \in \Sigma(q)$. Hence, for $c \in \Sigma(q)$, convexity and continuity (and piecewise linearity) of $f_{q,c}$ follow from proposition 4.1.8. Now, if c and c' are any two positive real numbers, it is clear that :

$$|f_{q,c}(\rho) - f_{q,c'}(\rho)| \leq q \cdot |c - c'| \qquad \text{for every } \rho \in \mathbb{R}_{\geq 0}.$$

Since $\Sigma(q)$ is dense in $\mathbb{R}$, it follows easily that $f_{q,c}$ is convex and continuous for every $c \in \mathbb{R}$. Similarly, for $q, q' \in \mathbb{R}_{\geq 0}$, we may bound the difference $|f_{q,c} - f_{q',c}|$ in terms of $|q - q'|$, on every bounded subset of $\mathbb{R}_{\geq 0}$; since $\mathbb{S}$ is dense in $\mathbb{R}_{\geq 0}$, we deduce continuity and convexity of $f_{q,c}$ for every $q \in \mathbb{R}_{\geq 0}$ and $c \in \mathbb{R}$. Next, for given $\rho_0 := -\log r_0 \geq 0$, let $\beta_1, \dots, \beta_k$ be the breaks of F_{r_0}, so that we have the break decomposition $F_{r_0} = F_{r_0}(\beta_1) \oplus \cdots \oplus F_{r_0}(\beta_k)$. Set $m_j := \mathrm{rk}_\Lambda F_{r_0}(\beta_j)$ for every $j \leq k$. We may find a segment $[\rho_0, \rho_1]$, and for every $i \leq d$, an integer $j_i \leq k$ such that :

$$f_i(\rho) = f_i(\rho_0) + (\rho - \rho_0) \cdot \beta^\natural_{j_i} \qquad \text{for every } \rho \in [\rho_0, \rho_1].$$

It follows easily that there exists $\rho_2 \in (\rho_0, \rho_1]$ such that :

$$f_{q,c}(\rho) = (qa + b) \cdot \rho + c' \qquad \text{for every } \rho \in [\rho_0, \rho_2] \tag{4.2.23}$$

where $a := \max\{i \leq d \mid -\log \beta_{j_i} \leq (q\rho_0 - c) + q\varepsilon\}$ (notation of (2.2.15)), and :

$$b := \sum_{j > j_a}^{k} m_j \beta^\natural_j \qquad c' := ca + \sum_{j > j_a}^{k} m_j(\beta^\flat_j - \rho_0 \beta^\natural_j).$$

This shows the piecewise linearity of $f_{q,c}$. We deduce as well that $b \in \mathbb{Z}$, since each term $m_j\beta^\natural_j$ is the Swan conductor of the Galois module $F_{r_0}(\beta_j)$ (denoted $\alpha(F_{r_0}(\beta_j))$ in [31, §8]).

This shows that (i) and (ii) hold. Moreover, (4.2.23) also easily implies (v). Assertion (iii) is already known for every pair (q, c) with $q \in \mathbb{S}$ and $c \in \Sigma(q)$ (theorem 4.2.2(i)). Next, if $c' \in \mathbb{R}$ is arbitrary, since the distance between $f_{q,c}$ and $f_{q,c'}$ is bounded, and $f_{q,c'}$ is convex and piecewise linear, it is easy to deduce that $f_{q,c'}$ is also eventually linear. Finally, if $q' \leq q$ is any positive real number, it is clear that $f_{q',c'} \leq f_{q,c'}$; since $f_{q,c'}$ is eventually linear and $f_{q',c'}$ is convex, it follows that right derivative $\rho \mapsto df_{q',c'}/dt(\rho^+)$ is non-decreasing and bounded; but from (ii) we see that the set of possible slopes for $f_{q'c'}$ does not admit accumulation points, hence the right derivative of $f_{q',c'}$ must be eventually constant. This concludes the proof of (iii).

Assertion (iv) is clear from (ii). ◊

Claim 4.2.22(iii) says that, for every $q \in \mathbb{R}_{\geq 0}$ and $c \in \mathbb{R}$, the limit :

$$s(q) := \lim_{\rho \to +\infty} f_{q,c}(\rho)/\rho \tag{4.2.24}$$

exists and is a rational number independent of c, whose denominator divides the denominator of q. Now, suppose $q \in \mathbb{S}$, $c \in \Sigma(q)$ and $c' > c$ is some real number; one sees easily that

$$\mathsf{sw}^\natural(F \otimes_\Lambda \mu_s^* \mathscr{L}(q), r^+) \geq \mathsf{sw}^\natural(F \otimes_\Lambda \mu_{s'}^* \mathscr{L}(q), r^+)$$

for every $r \in (0, 1]$ and every $s, s' \in K^+$ with $\log |s| = c$ and $\log |s'| = c'$. It follows that

$$\beta(F, q) = \sup\{\mathsf{sw}^\natural(F \otimes_\Lambda \mu_s^* \mathscr{L}(q), 0^+) \mid s \in K^+ \text{ and } \log |s| \in \Sigma(q)\} = s(q)$$

for every $q \in \mathbb{S}$.

Claim 4.2.25. The function $\mathbb{Q}_{\geq 0} \to \mathbb{R} : q \mapsto \beta(F, q)$ is non-decreasing.

Proof of the claim. It suffices to show that, if $q' < q < q''$ with $q', q'' \in \mathbb{S}$ and $q \in \mathbb{Q}$, then $\beta(F, q') \leq \beta(F, q) \leq \beta(F, q'')$. However, choose $s' \in K^+ \setminus \{0\}$ such that $c(q', s') \in \Sigma(q')$; from lemma 4.2.6 we see that there exists $\rho_0 \in \mathbb{R}$ such that $\Delta(\mathscr{L}(q)) \cap ([\rho_0, +\infty) \times \mathbb{R}_{\geq 0})$ is the graph of a linear map. We may then find $s \in K^\times$ such that $|s| < |s'|$ and such that $\Delta(\mu_s^* \mathscr{L}(q)) \cap$

$([\rho_0, +\infty) \times \mathbb{R}_{\geq 0}) \cap \Delta(F)$ is a countable subset. Since $\Delta(\mu_s^* \mathscr{L}(q))$ lies above $\Delta(\mu_{s'}^* \mathscr{L}(q'))$ in the region $[\rho_0, +\infty) \times \mathbb{R}_{\geq 0}$, an argument as in the foregoing shows that $\delta_{F \otimes \mu_s^* \mathscr{L}(q)}(\rho) > \delta_{F \otimes \mu_{s'}^* \mathscr{L}(q')}(\rho)$ for $\rho \geq \rho_0$. But since $q' \in \mathbb{S}$, we have seen that the slope of $\delta_{F \otimes \mu_{s'}^* \mathscr{L}(q')}$ equals $\beta(F, q')$, hence $\beta(F, q) \geq \beta(F, q')$, as required. The proof of the other inequality is similar, and shall be left to the reader. $\Diamond$

From claim 4.2.22(v) we deduce that s is a non-decreasing function. Since s and $\beta(F, \cdot)$ agree on the dense subset $\mathbb{S}$, they must coincide for all $q \in \mathbb{Q}_{\geq 0}$. Combining with claim 4.2.22(iv), this proves assertion (i).

(ii): From the definition of $f_{q,c}$, it is obvious that $f_{q,0}(\rho) \geq q\rho \cdot \mathrm{rk}_\Lambda F$ for every $\rho \in \mathbb{R}_{\geq 0}$, hence $s(q) \geq q \cdot \mathrm{rk}_\Lambda F$. Furthermore, since $\sum_{i=1}^d f_i(\rho) = \delta_F(\rho)$ is a convex function which is eventually linear of slope $q_0 := \mathrm{sw}^\natural(F, 0^+)$, one sees easily that there exists $c \in \mathbb{R}$ such that :

$$f_i(\rho) \leq q_0 \rho + c \qquad \text{for every } \rho \in \mathbb{R}_{\geq 0} \text{ and every } i \leq d.$$

Hence $f_{q,0}(\rho) = q\rho \cdot \mathrm{rk}_\Lambda F$ for every $q > q_0$, provided ρ is large enough. Consequently $s(q) = q \cdot \mathrm{rk}_\Lambda F$ for every $q > q_0$, so (ii) holds.

(iii): Claim 4.2.22(v) implies that the function $q \mapsto s(q)$ is convex and non-decreasing. Next, if $q.q'$ are any two positive real numbers, it is clear that $|s(q) - s(q')| \leq d \cdot |q - q'|$, so the mapping s is also continuous. Moreover, the convexity of s implies that the right derivative $ds/dt(\rho^+)$ exists for every $\rho \in \mathbb{R}_{>0}$ and is non-decreasing, and s is a primitive of its right derivative.

Claim 4.2.26. $ds/dt(\rho^+) \in \mathbb{Z}$ for every $\rho \in \mathbb{Q}$.

Proof of the claim. Write $\rho = a/b$ with relatively prime positive integers a, b. By definition, we have :

$$\frac{ds}{dt}(\rho^+) = \lim_{n \to +\infty} bn \cdot \left\{ s\left(\rho + \frac{1}{bn}\right) - s(\rho) \right\}. \tag{4.2.27}$$

Now, if we let n run over the positive integers, the right-hand side of (4.2.27) is the limit of a sequence of integers, since the denominators of both $(\rho + 1/(bn))$ and $s(\rho)$ divide bn. $\Diamond$

We deduce from claim 4.2.26 that the right derivative of s is a non-decreasing step function (constant on segments of the form $[a, b)$). Hence s is piecewise linear with integral slopes, which concludes the proof of (iii) and of the theorem. $\square$

4.2.28. Let F be as in theorem 4.2.19. The idea is that the graph of $\beta(F, \cdot)$ should be the Newton polygon associated to the sought break decomposition of the stalk F_0 of F over the missing point $\eta(0)$ (see (4.2.10)). According to this picture, the breaks of F_0 are the values $q_i \in \mathbb{R}_{>0}$ such that $ds/dt(q^-) \neq ds/dt(q^+)$ (where s is defined as in (4.2.24)); naturally we call these the *break points* of $\beta(F, \cdot)$. The first observation is that there are only finitely many break points, and all of them are rational; indeed, this is a straightforward consequence of theorem 4.2.19. Let $0 < q_1 < q_2 < \cdots < q_n$ be these break points, and set $q_0 := 0$. Since $\beta(F, \cdot)$ is piecewise linear and non-negative, we may find unique $\mu_0, \mu_1, \ldots, \mu_n \geq 0$ such that :

$$\beta(F, q) = \sum_{j=0}^n \mu_j \cdot \max(q_j, q) \qquad \text{for every } q \in \mathbb{Q}_{\geq 0}. \tag{4.2.29}$$

Indeed, by deriving both sides of (4.2.29), we find :

$$\sum_{j=0}^i \mu_j = \left. \frac{d\beta(F, q)}{dq} \right|_{q = q_i^+} \qquad \text{for every } j \leq n. \tag{4.2.30}$$

And since $\beta(F, \cdot)$ has integer slopes (theorem 4.2.19(iii)), we deduce that $\mu_j \in \mathbb{N}$ for every $j \leq n$. For every $j \leq n$, the integer μ_j should be nothing else than the rank of the direct factor

of F_0 which is pure of break q_j. This is borne out by the identity :

$$\mathrm{rk}_\Lambda F = \sum_{j=0}^{n} \mu_j$$

which holds, since $\beta(F,\cdot)$ is eventually linear of slope $\mathrm{rk}_\Lambda F$ (theorem 4.2.19(ii)). For this reason, we shall say that μ_i is the *multiplicity* of the break q_i, for every $i = 0, \dots, n$. Now, let :

$$0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_d$$

be the unique sequence of rational numbers in which the value q_i appears with multiplicity μ_i, for every $i = 0, \dots, n$. We have :

Theorem 4.2.31. *Keep the notation of* (4.2.28)*, and let* $d := \mathrm{rk}_\Lambda F$*. Then there exist* $\rho_0 \geq 0$*, and real numbers* $c_1, c_2, \dots, c_d$ *such that :*

$$f_i(\rho) = \tau_i \cdot \rho + c_i \qquad \textit{for every } \rho \geq \rho_0 \textit{ and every } i = 1, \dots, d$$

where $f_1, f_2, \dots, f_d : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *are defined as in* (4.2.12)*.*

Proof. Recall (claim 4.2.22) that $f_{q,0}(\rho) := \sum_{i=1}^{d} \max(f_i(\rho), q\rho)$ for every $\rho \in \mathbb{R}_{\geq 0}$. For every $\rho, y \in \mathbb{R}_{\geq 0}$, let us set :

$$I_{\rho,y} := \{i \in \mathbb{N} \mid 1 \leq i \leq d \ \text{ and } \ f_i(\rho) \leq y\} \qquad N_{\rho,y} := \sharp I_{\rho,y}$$

(where, as usual, for any set I, we denote by $\sharp I$ the cardinality of I). Let $q' > q$ be any two real numbers; we may compute :

$$f_{q',0}(\rho) - f_{q,0}(\rho) = q'\rho N_{\rho,q'\rho} - q\rho N_{\rho,q\rho} - \sum_{i \in J} f_i(\rho) \qquad \text{for every } \rho \geq 0$$

where :

$$J := \{i \in \mathbb{N} \mid 1 \leq i \leq d \ \text{ and } \ q\rho < f_i(\rho) < q'\rho\}.$$

It follows easily that :

$$g(q, q', \rho) := \frac{f_{q',0}(\rho) - f_{q,0}(\rho)}{(q' - q)\rho} \in [N_{\rho,q\rho}, N_{\rho,q'\rho}] \qquad \text{for every } \rho > 0. \tag{4.2.32}$$

Now, let $q > 0$ be any real number which is not a break point for $\beta(F,\cdot)$; let $k \in \{0, 1, \dots, n\}$ be the largest integer such that $q_k < q$. If $k = n$, set $q_{n+1} := q+1$, so that in any case $q \in (q_k, q_{k+1})$. We notice that the function s defined as in (4.2.24) is linear on the interval $[q_k, q_{k+1}]$, since the proof of theorem 4.2.19 shows that s agrees with $\beta(F,\cdot)$ on $\mathbb{Q}_{\geq 0}$. Especially :

$$\lim_{\rho \to +\infty} g(q, q_{k+1}, \rho) = \lim_{\rho \to +\infty} g(q_k, q, \rho) = \left. \frac{d\beta(F,q)}{dq} \right|_{q=q_k^+}.$$

But recall that the slopes of $\beta(F,\cdot)$ are integers; therefore, combining with (4.2.32) we deduce :

$$\left. \frac{d\beta(F,q)}{dq} \right|_{q=q_k^+} \in [N_{\rho,q_k\rho}, N_{\rho,q\rho}] \cap [N_{\rho,q\rho}, N_{\rho,q_{k+1}\rho}] = \{N_{\rho,q\rho}\} \quad \text{for all large } \rho. \tag{4.2.33}$$

The meaning of (4.2.33) is that, if q is not a break, then the points $(\rho, f_i(\rho))$ tend to "move away" from the line $\{(x,y) \mid qx = y\}$; indeed, (4.2.33) shows that if $q' \in (q_k, q)$ is any other real number, then $I_{\rho,q\rho} = I_{\rho,q'\rho}$ provided ρ is large enough. Fix any $\varepsilon > 0$ such that :

$$2\varepsilon < \min\{q_{k+1} - q_k \mid k = 0, \dots, n-1\}$$

and set :

$$J_k(\rho) := I_{\rho,(q_k+\varepsilon)\rho} \setminus I_{\rho,(q_k-\varepsilon)\rho} \qquad \text{for every } k \leq n \text{ and every } \rho \geq 0.$$

Notice that, since the functions f_i are continuous (lemma 4.2.13), each set $J_k(\rho)$ will be eventually independent of ρ (*i.e.* for large values of ρ), and we shall therefore denote it simply by J_k. Summing up, so far we have exhibited a natural partition :

$$\{1, 2, \dots, d\} = J_0 \amalg J_1 \amalg \cdots \amalg J_n$$

such that, for every $k \leq d$, the values $T_k(\rho) := \{(\rho, f_i(\rho)) \mid i \in J_k\}$ "cluster" around a straight line of slope q_k. Explicitly, for every $\varepsilon > 0$ and for every large ρ, the points of $T_k(\rho)$ lie in the cone $C_\varepsilon(k) := \{(x, y) \in \mathbb{R}^2_{>0} \mid |y/x - q_k| < \varepsilon\}$. Next we show that, for every large ρ, the set $T_k(\rho)$ actually lies in a band of slope q_k and fixed bounded width. To this aim, for every $k = 0, 1, \dots, n$ and every $c \in \mathbb{R}$, set :

$$h_k(\rho) := \sum_{i \in J_k} f_i(\rho) \qquad h^*_{k,c}(\rho) := \sum_{i \in J_k} \max(f_i(\rho), q_k\rho - c) \qquad \text{for every } \rho \in \mathbb{R}_{\geq 0}.$$

Claim 4.2.34. (i) For every $k \leq n$ the following holds :

(i) the functions h_k and $h^*_{k,c}$ are eventually linear.
(ii) $\sharp J_k = \mu_k$.
(iii) $\lim\limits_{\rho \to +\infty} h_k(\rho)/\rho = q_k\mu_k = \lim\limits_{\rho \to +\infty} h^*_{k,c}(\rho)/\rho$.

Proof of the claim. Suppose $q \in (q_k, q_{k+1})$. We can then write :

$$f_{q,0}(\rho) = q\rho \cdot \sum_{t \leq k} \sharp J_t + \sum_{t=k+1}^{n} h_t(\rho) \qquad \text{for every sufficiently large } \rho.$$

Since the function $f_{q,0}(\rho)$ is eventually linear, we deduce that, for every $k \leq n$, the sum $\sum_{t=k+1}^n h_t$ is eventually linear, so the same holds for each term h_t. Let $C_t := \lim\limits_{\rho \to +\infty} h_t(\rho)/\rho$. In view of (4.2.24) we deduce :

$$s(q) = q \cdot \sum_{t \leq k} \sharp J_t + \sum_{t=k+1}^{n} C_t \qquad \text{for every } q \in (q_k, q_{k+1}) \text{ and every } k \leq n.$$

Now suppose that $q' \in (q, q_{k+1})$. Taking into account (4.2.30), we find :

$$(q' - q) \cdot \sum_{t \leq k} \sharp J_t = s(q') - s(q) = (q' - q) \cdot \sum_{j \leq k} \mu_j$$

from which (ii) follows easily, arguing by induction on k. Finally, on the one hand we know that h_k is eventually linear; on the other hand, for every $\varepsilon > 0$, each of its summands f_i (for $i \in J_k$) is eventually contained in the cone $C_\varepsilon(k)$, so assertion (iii) for h_k follows easily from (ii). Next, we look at the identity :

$$f_{q_k,c}(\rho) = (q_k\rho - c) \cdot \sum_{t<k} \mu_t + \sum_{t=k+1}^{n} h_t(\rho) + h^*_{k,c}(\rho)$$

which holds for every $k \leq n$ and every large enough ρ, in view of (ii). Since $f_{q_k,0}$ and $h_{k+1}, \dots, h_n$ are eventually linear functions, we see that the same holds for $h^*_{k,c}$, for every $k \leq n$. This shows (i), and also the remaining assertion (iii) for h^*_k follows easily. $\diamondsuit$

We now fix $k \in \{0, 1, \dots, n\}$, and write just q, μ J, h and h^*_c instead of q_k, μ_k, J_k, h_k, $h^*_{k,c}$.

Claim 4.2.35. For every $i \in J$, the function

$$\rho \mapsto |f_i(\rho) - q\rho|$$

is bounded.

Proof of the claim. It follows easily from claim 4.2.34 that both functions :

$$\sum_{i\in J}\{\max(f_i(\rho),q\rho)-q\rho\} \quad \text{and} \quad \sum_{i\in J}\{\max(f_i(\rho),q\rho)-f_i(\rho)\}$$

are eventually constant. Since these summands are always non-negative, we deduce that, for every $i\in J$, the terms :

$$\max(f_i(\rho),q\rho)-q\rho \quad \text{and} \quad \max(f_i(\rho),q\rho)-f_i(\rho)$$

are bounded, which is the claim. ♢

Claim 4.2.36. For every $i\in J$ there exists $a_i\in\mathbb{R}$ such that :

$$\lim_{\rho\to+\infty} f_i(\rho)-q\rho=a_i.$$

Proof of the claim. Say that $J=\{i_0,\dots,i_0+\mu-1\}$. We prove, by induction on t, that a_{i_0+t} with the desired property exists for every $t<\mu$. For $t<0$, there is nothing to prove. Next, suppose that $t\geq 0$ and that the assertion is already known for every integer $<t$; we set :

$$g(\rho):=\sum_{i=i_0+t}^{i_0+\mu-1} f_i(\rho) \quad g^*_c(\rho):=\sum_{i=i_0+t}^{i_0+\mu-1}\max(f_i(\rho),q\rho-c) \qquad \text{for every } \rho\in\mathbb{R}_{\geq 0} \text{ and } c\in\mathbb{R}.$$

Using the inductive assumption, and claim 4.2.34, we see that there exists $C\in\mathbb{R}$ with :

$$\lim_{\rho\to+\infty} g(\rho)-\rho q(\mu-t)=C. \tag{4.2.37}$$

Set :

$$a:=\liminf_{\rho\to+\infty} f_{i_0+t}(\rho)-q\rho \qquad b:=\limsup_{\rho\to+\infty} f_{i_0+t}(\rho)-q\rho.$$

Notice that :

$$a\geq a_{i_0},\dots,a_{i_0+t-1} \tag{4.2.38}$$

since $f_i\leq f_{i+1}$ for every $i=1,\dots,d-1$. Suppose $a<b$, pick $x\in(a,b)$ and set $c:=-x$; in view of (4.2.38), we have :

$$h^*_c(\rho)=t(q\rho-c)+g^*_c(\rho) \qquad \text{for every large enough } \rho.$$

Then claim 4.2.34 implies that g^*_c is eventually linear of slope $q(\mu-t)$. Combining with (4.2.37), we deduce that there exists $C'\in\mathbb{R}$ such that :

$$\lim_{\rho\to+\infty} g^*_c(\rho)-g(\rho)=C'. \tag{4.2.39}$$

However, due to our choice of x, for every $\rho\geq 0$ and every $\varepsilon>0$ we may find $\rho',\rho''\geq\rho$ such that :

$$g^*_c(\rho')=g(\rho') \qquad \text{and} \qquad g^*_c(\rho'')-g(\rho'')>x-a+\varepsilon$$

which contradicts (4.2.39). Hence $a=b$, and the common value is a real number, due to claim 4.2.35. This concludes the inductive step. ♢

To conclude the proof of the theorem, we shall show that the function f_{i_0+t} is eventually linear, whenever $i_0+t\in J=\{i_0,\dots,i_0+\mu-1\}$. We shall proceed by induction on t. If $t<0$, there is nothing to prove. Hence, suppose that the assertion is known for every integer $<t$. Set $a:=a_{i_0+t}$, where $a_{i_0},\dots,a_{i_0+\mu-1}$ are the real numbers whose existence is ensured by claim 4.2.36. Let $J(a):=\{i\in J\mid a_i=a\}$; we shall show simultaneously that all the functions f_i with $i\in J(a)$ are eventually linear, hence we may suppose that i_0+t is the smallest element of $J(a)$. In this case, using the inductive assumption and claim 4.2.34, we see that both functions

g and g^*_{-a} introduced in the proof of claim 4.2.36 are eventually linear. Moreover, it is also clear that :

$$\lim_{\rho\to+\infty} g^*_{-a}(\rho) - g(\rho) = 0.$$

It follows that $g(\rho) = g^*_{-a}(\rho)$ for every large enough ρ, hence

$$f_i(\rho) \geq qp + a \qquad \text{for every } i \in J(a) \text{ and every large } \rho. \tag{4.2.40}$$

On the other hand, let i_1 be the largest element of $J(a)$; if $i_1 < i_0 + \mu - 1$, choose $b \in (a, a_{i_1+1})$; otherwise, set $b := a_{i_1} + 1$. In either case, we may write :

$$h^*_{-b}(\rho) = i_1(q\rho + b) + \sum_{i>i_1} f_i(\rho) \qquad \text{for every large enough } \rho.$$

Hence $\sum_{i>i_1} f_i$ is eventually linear, and therefore the same holds for $g - \sum_{i>i_1} f_i = \sum_{i\in J(a)} f_i$. Clearly :

$$\lim_{\rho\to+\infty} \sum_{i\in J(a)} (f_i(\rho) - q\rho - a) = 0.$$

Combining with (4.2.40), we deduce the contention. □

4.2.41. Let $(\Gamma_0, \leq)$ be the abelian group $\mathbb{Q} \times \Gamma_K$, endowed with the ordering such that :

$$(q, c) \leq (q', c') \qquad \text{if and only if either } q' < q \text{ or else } q = q' \text{ and } c \leq c'.$$

(This is the lexicographic ordering, except that the ordering on $\mathbb{Q}$ is the reverse of the usual one.) For given $r \in \Gamma^+$, let Γ_r be the value group of the valuation $|\cdot|_{\eta(r)}$. The mapping :

$$\Gamma_0 \to \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_r \quad : \quad (q, c) \mapsto c \cdot r^q \cdot (1-\varepsilon)^q$$

is an isomorphism of groups which does not respect the orderings (indeed, the ordering on Γ_0 induced by this isomorphism is also lexicographic, but the two factors $\mathbb{Q}$ and Γ_K are swapped). Nevertheless, we may interpret theorem 4.2.31, by saying that the "missing stalk F_0" admits a break decomposition which is naturally indexed by elements of Γ_0^+. More precisely, we have :

Theorem 4.2.42. *Let F be as in theorem* 4.2.19*. Then there exist $r_0 \in \Gamma_K^+$, a connected open subset $U \subset \mathbb{D}(1)^*$ and a decomposition :*

$$F_{|U} = \bigoplus_{(q,c)\in\Gamma_0^+} M(q, c)$$

where each summand $M(q, c)$ is a locally constant Λ-module on $U_{\text{ét}}$, such that :

(i) *$U \cap \mathbb{D}(\varepsilon) \neq \varnothing$ for every $\varepsilon \in \Gamma_K^+$.*
(ii) *For every $r \leq r_0$, we have $\eta(r) \in U$ and :*

$$M(q, c)_{\eta(r)} = F_r(c \cdot r^q \cdot (1-\varepsilon)^q).$$

Proof. Set $E := \mathscr{E}nd_\Lambda(F)$, the sheaf of Λ-linear endomorphisms of F. We have to exhibit $U \subset \mathbb{D}(1)^*$ fulfilling (i), and for each $(q, c) \in \Gamma_0^+$, a projector $\pi \in E(U)$ such that

$$(\mathrm{Im}\,\pi)_{\eta(r)} = F_r(c \cdot r^q \cdot (1-\varepsilon)^q).$$

By theorem 4.2.31, we may find $\rho_0 \geq 0$ such that the functions f_i are linear on the half-line $[\rho_0, +\infty)$; up to replacing ρ_0 by a larger real number, we may achieve that, for every $i, j \leq d$, the graphs of the functions f_i, f_j are either disjoint or equal. Say that $\rho_0 = -\log r_0$. The open subset U shall be constructed by removing from $\mathbb{D}(r_0)^*$ infinitely many closed discs. Indeed, suppose $r \leq r_0$, and let $\beta_1(r), \dots, \beta_k(r)$ be the breaks of F_r; we may assume that

$c \cdot r^q \cdot (1-\varepsilon)^q = \beta_1(r)$. From our choice of ρ_0, it follows that $\beta_i(r)^\flat = \beta_j(r)^\flat$ if and only if $i = j$; in other words, the decomposition (4.1.16) is the same as the decomposition

$$F_{\eta(r)^\flat} = \bigoplus_{\alpha \in \Gamma_K^+} F_r^\flat(\alpha)$$

(notation of (4.2.11)). Hence, the stalk $E_{\eta(r)^\flat}$ contains a projector $\pi_{\eta(r)^\flat}$ that cuts out the summand $F_r(\beta_1(r))$. However, since F is locally constant on $\mathbb{D}(1)^*_{\text{ét}}$, the same holds for E; especially, E is overconvergent, in the sense of [30, Def.8.2.1]. Let :

$$\mathbb{D}(1)^*_{\text{ét}} \xrightarrow{\mu} \mathbb{D}(1)^* \xrightarrow{\nu} \mathbb{D}(1)^*_{\text{p.p}}$$

be the natural morphisms of sites, where $\mathbb{D}(1)^*_{\text{p.p}}$ denotes the topological space $\mathbb{D}(1)^*$ endowed with its partially proper topology ([30, Def.8.1.3]). On the one hand, using [30, Prop.1.5.4], we deduce that the natural map :

$$(\mu_* E)_{\eta(r)^\flat} \to E_{\eta(r)^\flat}$$

is a bijection; on the other hand, according to [30, Prop.8.1.4(a)], the counit of adjunction $\nu^*\nu_*(\mu_* E) \to (\mu_* E)$ is an isomorphism, hence also the natural map

$$(\nu_*\mu_* E)_{\eta(r)^\flat} \to \mu_* E_{\eta(r)^\flat}$$

is bijective. Therefore, we may find a partially proper open neighborhood $V \subset \mathbb{D}(1)^*$ of $\eta(r)^\flat$, and a section $\pi_V \in E(V)$, such that $(\pi_V)_{\eta(r)^\flat} = \pi_{\eta(r)^\flat}$. Since E is locally constant and $\pi^2_{\eta(r)^\flat} = \pi_{\eta(r)^\flat}$, it follows that π_V is a projector in $E(V)$, and its stalk $(\pi_V)_{\eta(r)}$ cuts out the direct summand $F_r(\beta_1(r))$.

Next, for every $r' < r$, let $\mathbb{D}(r', r^-) := \bigcup_{r' < s < r} \mathbb{D}(r', s)$; by inspecting the proof of theorem 3.3.29 we see that, provided r' is sufficiently close to r, there exists a decomposition :

$$F_{|\mathbb{D}(r', r^-)} \simeq \bigoplus_{j=1}^k G_j$$

consisting of locally constant Λ-modules G_j on $\mathbb{D}(r', r^-)_{\text{ét}}$, such that :

$$(G_j)_{\eta(s)} = F_s(\beta_j(s)) \qquad \text{for every } j \le k \text{ and every } s \in [r', r).$$

Thus, we may find a unique projector $\pi_{\mathbb{D}(r', r^-)} \in E(\mathbb{D}(r', r^-))$ that cuts out the direct summand G_1. Finally, on the intersection $V \cap \mathbb{D}(r', r^-)$, we must have $\pi_V = \pi_{\mathbb{D}(r', r^-)}$, hence we obtain a section $\pi_{W(r)}$ of E on the open subset $W(r) := V \cup \mathbb{D}(r', r^-)$. Up to removing some small closed subset, we may assume that $W(r)$ is of the form $\mathbb{D}(r', r) \setminus \bigcup_{i=1}^m \mathbb{E}(a_i, \rho_i)$, where $\mathbb{E}(a_i, \rho_i)$ denotes the closed disc with center $a_i \in \mathbb{D}(r, r)$ and radius $\rho_i < r$. By a standard compactness argument, we see that there exists a sequence $(r_n \mid n \in \mathbb{N})$ of elements of Γ_K, with $r_n \le r_0$ for every $n \in \mathbb{N}$, and $\lim_{n \to +\infty} r_n = 0$, such that $U := \bigcup_{n \in \mathbb{N}} W(r_n)$ contains the subset $\{\eta(s) \mid s \in \Gamma_K;\ \ s \le r_0\}$. Clearly, this open subset U will do. $\square$

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